

Lenses and Asymptotic Midpoint Uniform Convexity

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Abstract

We introduce a new isometric geometric property, namely Asymptotic Midpoint Uniform Convexity and investigate its connection to similar asymptotic properties.

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1. Introduction and some preliminary notions and results

Geometric properties of the unit ball of a Banach space are widely used in applications, from nonlinear analysis to best approximation theory. In 1987 S.

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Rolewicz [Ro1, Ro2] introduced the following property called by him (β) -property: a given real Banach space $(X, \|\cdot\|)$ with closed unit ball B_X is said to possess the (β) -property if for every $\varepsilon > 0$ there is $\delta > 0$ such that for each $x \in X$ with $1 < \|x\| < 1 + \delta$ the set $\text{conv}(x \cup B_X) \setminus B_X$ has Kuratowski measure of noncompactness less than ε . Here, as usual, for a set $A \subset X$, the symbol $\text{conv}(A)$ has the meaning of convex hull of the set A and the Kuratowski measure of noncompactness of the set A , denoted by $\alpha(A)$, is the infimum of all $\varepsilon > 0$ such that A can be covered by a finite number of sets of diameters less than ε . Equivalently, one says that the norm in X possesses property (β) . One of the motivations for the introduction of (β) -property was the fact that if, in the above definition, we replace the Kuratowski measure of noncompactness by the diameter of the corresponding set we obtain a characterization of the well-known property of uniform convexity of the norm in X . Another important point is that property (β) implies that the space is reflexive.

Recently, Lima and Randrianarivony [LR] used property (β) in its equivalent form given by Kutzarova [Ku] to answer a ten-year old question of Bates, Johnson, Lindenstrauss, Preiss and Schechtman [BJLPS] about uniform quotients. Independently, Revalski and Zhivkov [ReZh] defined the notion of compact uniform convexity in connection to the study of metric projections. We established in [DKRReZh] that the compact uniform convexity coincides with property (β) of Rolewicz.

In [KuPa], another isometric characterization of property (β) , in terms of certain kind of lenses, was given. Let $y \in B_X$ and define,

$$\text{Lens}(y, \sigma) := B[y, 1 - \|y\| + \sigma] \setminus B[0, 1], \quad 0 < \|y\| < 1, \sigma \in (0, 2\|y\|),$$

where, as usual, $B[z, r]$, $z \in X$ and $r > 0$, stands for the closed ball centered at z and with radius r .

With this definition in hand we recall the following result.

Theorem 1.1. ([KuPa]) *Let X be a Banach space. If the norm in X has the (β) -property, then for every $0 < t < 1$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that for every x , $\|x\| = 1$, we have $\alpha(\text{Lens}(tx, \delta)) < \varepsilon$. Conversely, if for some $t \in (0, 1)$ we have $\alpha(\text{Lens}(tx, \delta)) \rightarrow 0$ as $\delta \rightarrow 0$ uniformly in x , $\|x\| = 1$, then X has the (β) -property.*

In [DKRReZh] we have given a further qualitative version of the latter result. Let us mention that, if in the above theorem we replace the Kuratowski measure of noncompactness by the diameter of the corresponding lenses, we obtain a characterization of uniform convexity of the norm in X (c.f. [St], [KuPa]).

A different type of lenses, again related to the study of uniform-type properties of the norm, is considered by Laakso [La] and by Tyson and Wu [TW]. Let $x \in X$ be such that $\|x\| = 1$ and let $\delta \in (0, 1)$. Define,

$$\text{lens}(x, \delta) := B[x, 1 + \delta] \cap B[-x, 1 + \delta].$$

It is proved in [La] (see also [TW]) that the condition $\text{diam}(\text{lens}(x, \delta)) \rightarrow 0$ as $\delta \rightarrow 0$ uniformly in $x, \|x\| = 1$, isometrically characterizes uniform convexity of the norm in X .

It is natural to ask the question what kind of geometric property one obtains if one replaces the diameter of the lenses by their Kuratowski index of noncompactness in Laakso's characterization of uniform convexity. It turns out that the new property is not equivalent to property (β) . In fact, it will be shown in this article that this new property is implied by asymptotic uniform convexity and is therefore satisfied by the space l_1 . In particular, this new property does not imply property (β) isomorphically nor even reflexivity.

2. Asymptotic Midpoint Uniform Convexity

Let $(X, \|\cdot\|)$ be a real Banach space and S_X denotes its unit sphere. We begin this section by the following characterization:

Theorem 2.1. *Let X be a Banach space. Then the following are equivalent:*

- (i) $\alpha(\text{lens}(x, \delta)) \rightarrow 0$ as $\delta \rightarrow 0$ uniformly in $x \in S_X$;
- (ii) *For every $0 < t < 1$ there is $\delta(t) > 0$ such that for every $x \in S_X$ there exists a closed subspace $Y \subset X$ with $\text{codim}(Y) < +\infty$ satisfying the property:*

$$\inf_{y \in S_Y} \max(\|x + ty\|, \|x - ty\|) \geq 1 + \delta(t).$$

Proof. (i) $\Rightarrow$ (ii) Assume that (ii) fails. Then there exists a $t > 0$ such that for every $\delta > 0$ there is $x = x_\delta \in S_X$ such that for every closed subspace Y of finite codimension in X there exists a $y \in S_Y$ such that

$$\max(\|x + ty\|, \|x - ty\|) < 1 + \delta.$$

Choose inductively a sequence $(y_n)_{n=1}^\infty \subset S_X$, with $y_n \in Y_{n-1} := \cap_{i=1}^{n-1} \ker f_i$, where $f_i \in S_{X^*}$ is a norming functional for y_i and $\ker f_i$ denotes the kernel of f_i such that

$$\max(\|x + ty_n\|, \|x - ty_n\|) < 1 + \delta.$$

Thus the sequence $(ty_n)_{n=1}^\infty$ is contained in $\text{lens}(x, \delta) = B[x, 1+\delta] \cap B[-x, 1+\delta]$. Since $\|y_m - y_n\| \geq f_m(y_m - y_n) = 1$ for $n > m$, we have

$$\alpha(\text{lens}(x, \delta)) \geq t$$

which contradicts (i).

Let us prove now that (ii) $\Rightarrow$ (i). To this end we will follow an argument from [R].

Let $t \in (0, 1)$. We can find by (ii) a $\delta(t) > 0$ such that the properties in the assertion (ii) are satisfied. Let $x \in S_X$ and let Y be a closed subspace with finite codimension for which we will have:

$$\inf_{y \in S_Y} \max(\|x + ty\|, \|x - ty\|) \geq 1 + \delta(t).$$

By convexity we also have

$$\inf_{y \in Y, \|y\| \geq 1} \max(\|x + ty\|, \|x - ty\|) \geq 1 + \delta(t).$$

In fact, if $y \in Y$ and $z = \lambda y$ for some $\lambda \geq 1$, then $x + ty = \frac{1}{\lambda}(x + tz) + (1 - \frac{1}{\lambda})x$. Consequently, $\|x + ty\| \leq \frac{1}{\lambda}\|x + tz\| + (1 - \frac{1}{\lambda})$ so that $\|x + ty\| - 1 \leq \frac{1}{\lambda}(\|x + tz\| - 1) \leq \|x + tz\| - 1$. We now then have

$$\text{lens}(x, \delta(t)/2) \cap Y \subset B_Y[0, t],$$

where the subscript Y means the corresponding ball considered only in Y . Consider the set $\text{lens}(x, \delta(t)/2)$. It is a closed, convex, symmetric neighborhood of 0 in X , thus it may be viewed as a unit ball of an equivalent norm on X (with a constant of equivalence depending on t). Therefore, we can deduce from [JLPS, Lemma 2.13] that there exists a compact set $K \subset X$ such that

$$\text{lens}(x, \delta(t)/2) \subseteq K + 3 \text{lens}(x, \delta(t)/2) \cap Y.$$

Combining the last two inclusions we obtain

$$\text{lens}(x, \delta(t)/2) \subset K + 3 B_Y[0, t] \subset K + B[0, 3t],$$

which implies that the Kuratowski measure of noncompactness of $\text{lens}(x, \delta(t)/2)$ is at most $3t$. This completes the proof. ■

K. W. Anderson [A] studied in his Ph.D. dissertation the notion of *midpoint local uniform convexity* (MLUC), a notion which had been known earlier to G.

Lumer and M. M. Day. As given in [A], the norm in a Banach space is MLUC if for every $x \in S_X$ and every sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in S_X , we have that whenever $\|(x_n + y_n) - 2x\| \rightarrow 0$ as $n \rightarrow \infty$, it follows that $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. Proposition 5.3.26 in [M], as well as the proof following it, show that Anderson's definition is equivalent to the following: for every $x \in S_X$ we have that whenever $(x_n)_{n=1}^\infty$ is a sequence in X with both $\|x + x_n\| \rightarrow 1$ and $\|x - x_n\| \rightarrow 1$ as $n \rightarrow \infty$, it follows that $x_n \rightarrow 0$ as $n \rightarrow \infty$. The equivalent condition (ii) in the above theorem resembles an asymptotic uniform version of MLUC, which motivates us to give the following name of the new geometric property satisfying the conditions of Theorem 2.1.

Definition 2.2. *We say that the norm in a Banach space X is asymptotically midpoint uniformly convex (in short AMUC) if $\alpha(\text{lens}(x, \delta)) \rightarrow 0$ as $\delta \rightarrow 0$ uniformly in $x \in S_X$. The asymptotic midpoint uniform convexity modulus of X is defined for $t > 0$ as follows:*

$$\tilde{\delta}_X(t) := \inf_{x \in S_X} \sup_{\text{codim}(Y) < \infty} \inf_{y \in S_Y} [\max(\|x + ty\|, \|x - ty\|) - 1],$$

where Y denotes a closed linear subspace of X . (Note then that X is AMUC if and only if $\tilde{\delta}_X(t) > 0$ whenever $t > 0$.)

Recall (see for example [H]) that a Banach space $(X, \|\cdot\|)$ is said to have the *Kadec-Klee property* (the KK-property for short) if whenever $(x_n)_{n=1}^\infty \subset S_X$ and $x \in S_X$ with $x_n \rightarrow x$ weakly, it follows that $x_n \rightarrow x$ in norm. The Banach space $(X, \|\cdot\|)$ is said to have the *uniform Kadec-Klee property* (the UKK-property for short) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $(x_n)_{n=1}^\infty$ is a sequence in $B[0, 1]$ with $\|x_n - x_m\| \geq \varepsilon$ for all $n \neq m$, if $x_n \rightarrow x$ weakly for some element x , then it follows that $\|x\| \leq 1 - \delta$. It is an easy exercise to see that X has the UKK-property if and only if for every $t > 0$ there exists $\delta > 0$ such that for all $x \in S_X$ and all weakly null sequence $(x_n)_{n=1}^\infty$ in X , if $\|x_n\| \geq t$ for all $n \geq 1$, then it follows that $\limsup_{n \rightarrow \infty} \|x + x_n\| > 1 + \delta$. Using condition (ii) of Theorem 2.1 we obtain the following partial characterization of AMUC in terms of an analogue of the UKK-property.

Corollary 2.3. *Suppose X is AMUC. Then for all $t > 0$ there exists $\delta > 0$ such that for all $x \in S_X$ and for all weakly null sequences $(x_n)_{n=1}^\infty$ contained in S_X , we have*

$$\limsup_{n \rightarrow \infty} \max(\|x + tx_n\|, \|x - tx_n\|) > 1 + \delta.$$

Moreover, the converse holds if X does not contain ℓ_1 .

Proof. The first statement just follows from the fact that if Y has finite codimension then $d(x_n, Y) \rightarrow 0$ for every weakly null sequence $(x_n)_{n=1}^\infty$. For the converse statement, suppose that the given property of weakly null sequences holds and that X fails the AMUC property. Let t, x , and (for a given $\delta > 0$) $(y_n)_{n=1}^\infty$ be as in the proof of Theorem 2.1. By Rosenthal's ℓ_1 theorem [Ros], since X does not contain ℓ_1 , we may assume (by passing to a subsequence) that $(y_n)_{n=1}^\infty$ is weakly Cauchy. Let $z_n := y_{2n} - y_{2n-1}$ for $n \geq 1$. Then $(z_n)_{n=1}^\infty$ is weakly null, $\|z_n\| \geq 1$, and

$$\left\| x + \frac{t}{2} z_n \right\| \leq \frac{1}{2} (\|x + ty_{2n}\| + \|x - ty_{2n-1}\|) \leq 1 + \delta.$$

Similarly $\left\| x - \frac{t}{2} z_n \right\| \leq 1 + \delta$. Hence

$$\max \left(\left\| x + \frac{t}{2} \frac{z_n}{\|z_n\|} \right\|, \left\| x - \frac{t}{2} \frac{z_n}{\|z_n\|} \right\| \right) \leq 1 + \delta,$$

which is a contradiction. ■

The study of the asymptotic geometry of Banach spaces goes back to Milman [Mi] where, in particular, he defines the asymptotic uniform convexity and asymptotic uniform smoothness, although he uses different names and notation. The modern terminology was introduced in [JLPS]. Recall that the norm $\|\cdot\|$ in a Banach space X is asymptotically uniformly convex (AUC) if we have for all $t > 0$

$$\bar{\delta}_X(t) := \inf_{x \in S_X} \sup_{\text{codim}(Y) < \infty} \inf_{y \in S_Y} (\|x + ty\| - 1) > 0,$$

where Y denotes a closed linear subspace of X . The function $\bar{\delta}_X$ is called *the asymptotic uniform convexity modulus* of X .

In view of Theorem 2.1, clearly AUC implies AMUC isometrically. The converse is not true as the following theorem demonstrates.

Theorem 2.4. *There exists a Banach space whose norm is AMUC but not AUC.*

Our example will be a renorming of l_2 . Let $\|\sum_{n=1}^\infty a_n e_n\|_2 = (\sum_{n=1}^\infty a_n^2)^{1/2}$ be the usual norm in l_2 , where $(e_n)_{n=1}^\infty$ is the canonical basis. We introduce an equivalent norm $\|\cdot\|$ on l_2 by defining the unit ball by

$$B(l_2, \|\cdot\|) := \overline{\text{conv}}(B(l_2, \|\cdot\|_2) \cup \{\pm(e_1 + e_n)_{n=2}^\infty\}),$$

where for a set A , $\overline{\text{conv}}(A)$ represents the closure of the convex hull of A under the topology of $\|\cdot\|_2$. We gather the properties of $\|\cdot\|$ in the following lemma.

Lemma 2.5. *The space $(l_2, \|\cdot\|)$ and its dual space $(l_2, \|\cdot\|^*)$ satisfy the following properties.*

- (a) $\|\cdot\| \leq \|\cdot\|_2 \leq \sqrt{2} \|\cdot\|$;
- (b) $\|\cdot\|^*$ is given by

$$\left\| \sum_{n=1}^{\infty} a_n e_n^* \right\|^* = \max \left(\left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/2}, \max_{n \geq 2} |a_1 + a_n| \right);$$

- (c) $(e_n)_{n=1}^{\infty}$ is a bimonotone basis for $(l_2, \|\cdot\|)$, i.e. for every $x = \sum_{n=1}^{\infty} a_n e_n$ and every $1 \leq i \leq j$ we have $\|\sum_{n=i}^j a_n e_n\| \leq \|x\|$;
- (d) $\|\sum_{n=2}^{\infty} a_n e_n\| = \|\sum_{n=2}^{\infty} a_n e_n^*\|^* = \|\sum_{n=2}^{\infty} a_n e_n\|_2$;
- (e) if $a_1 \neq 0$ then $\|\sum_{n=2}^{\infty} a_n e_n^*\|^* < \|\sum_{n=1}^{\infty} a_n e_n^*\|^*$;
- (f) $\|e_1\| = \|e_1 + e_n\| = 1$ for all $n \geq 2$.

Proof. (a) This comes from the inclusions

$$B(l_2, \|\cdot\|_2) \subset B(l_2, \|\cdot\|) \subset \sqrt{2} B(l_2, \|\cdot\|_2).$$

- (b) For $x^* = \sum_{n=1}^{\infty} a_n e_n^*$, denote $N(x^*) = \max \left(\left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/2}, \max_{n \geq 2} |a_1 + a_n| \right)$. Let $x = \lambda_1 u + \sum_{j=2}^J \lambda_j \varepsilon_j (e_1 + e_{n_j})$, where $u \in B(l_2, \|\cdot\|_2)$, $\varepsilon_1, \dots, \varepsilon_J \in \{-1, +1\}$, and $\lambda_1, \dots, \lambda_J \in [0, 1]$ with $\sum_{j=1}^J \lambda_j = 1$. Then we have $|x^*(x)| \leq \lambda_1 |x^*(u)| + \sum_{j=2}^J \lambda_j |x^*(e_1 + e_{n_j})| \leq \sum_{j=1}^J \lambda_j N(x^*)$. Consequently we have $|x^*(x)| \leq N(x^*)$ for every $x \in \text{conv}(B(l_2, \|\cdot\|_2) \cup \{\pm(e_1 + e_n)_{n=2}^{\infty}\})$, and hence for every $x \in B(l_2, \|\cdot\|)$ by continuity of x^* . This gives $\|x^*\|^* \leq N(x^*)$.

Conversely, by the classical isometric duality between c (the space of all convergent sequences) and l_1 , between l_2 and itself, and between c_0^2 and l_1^2 , we can find $(v_n)_{n=2}^{\infty}$ with $\sum_{n=2}^{\infty} |v_n| = 1$, $(u_n)_{n=1}^{\infty}$ with $(\sum_{n=1}^{\infty} u_n^2)^{1/2} = 1$, and α, β with $|\alpha| + |\beta| = 1$ so that

$$\begin{aligned} \max_{n \geq 2} |x^*(e_1 + e_n)| &= \sum_{n=2}^{\infty} v_n x^*(e_1 + e_n), \\ \left(\sum_{n=1}^{\infty} |x^*(e_n)|^2 \right)^{1/2} &= \sum_{n=1}^{\infty} u_n x^*(e_n), \end{aligned}$$

and finally

$$N(x^*) = \alpha \sum_{n=1}^{\infty} u_n x^*(e_n) + \beta \sum_{n=2}^{\infty} v_n x^*(e_1 + e_n).$$

In other words, $N(x^*) = \lim_{K \rightarrow \infty} x^*(y_K)$ where $y_K = \alpha \sum_{n=1}^K u_n e_n + \beta \sum_{n=2}^K v_n (e_1 + e_n)$. Note that y_K does belong to $\text{conv}(B(l_2, \|\cdot\|_2) \cup \{\pm(e_1 + e_n)_{n=2}^{\infty}\})$ so that $x^*(y_K) \leq \|x^*\|^*$. By letting $K \rightarrow \infty$, we get $N(x^*) \leq \|x^*\|^*$.

- (c) We claim that any vector $y \in (l_2, \|\cdot\|)$ can be $\|\cdot\|$ -normed by a functional $y^* \in (l_2, \|\cdot\|^*)$ with $\text{supp}(y^*) \subset \text{supp}(y)$ (where $\text{supp}(v)$ denotes as usual the set of indices of the nonzero coordinates in the standard representation of v). In fact, if x^* is a functional such that $\|x^*\|^* = 1$ and $x^*(y) = \|y\|$, consider the functional y^* where $y^*(e_n) = x^*(e_n)$ if $n \in \text{supp}(y)$, and $y^*(e_n) = 0$ otherwise. Then $\|y\| = x^*(y) = y^*(y) \leq \|y^*\|^* \|y\|$ so that $\|y^*\|^* \geq 1$. But by the formula in (b), we necessarily have $\|y^*\|^* \leq \|x^*\|^* = 1$.

Now, let $x = \sum_{n=1}^{\infty} a_n e_n$, let $\mathbb{J}$ be any subset of $\mathbb{N}$, and set $y = \sum_{n \in \mathbb{J}} a_n e_n$. Let y^* be a functional with $\text{supp}(y^*) \subset \text{supp}(y)$ that $\|\cdot\|$ -norms y . Then

$$\|y\| = y^*(y) = y^*(x) \leq \|y^*\|^* \|x\| = \|x\|.$$

- (d) The last equality is clear from the formula (b) for $\|\cdot\|^*$. For the first equality, let $x = \sum_{n=2}^{\infty} a_n e_n$, and let x^* be a functional with $\text{supp}(x^*) = \text{supp}(x)$ that $\|\cdot\|_2$ -norms x . Then $\|x\|_2 = x^*(x) \leq \|x^*\|^* \|x\| = \|x^*\|_2 \|x\|$, where this last equality is because $\text{supp}(x^*) \subset [2, \infty)$. Since $\|x^*\|_2 = 1$, we get that $\|x\|_2 \leq \|x\|$. The reverse inequality comes from (a).
- (e) This easily comes from formula (b).
- (f) Consider the vector $e_1 + v$ where v is either 0 or e_n for some $n \geq 2$. Then $1 = e_1^*(e_1 + v) \leq \|e_1^*\|^* \|e_1 + v\|$. But $\|e_1^*\|^* = 1$ by formula (b), so that $1 \leq \|e_1 + v\|$. The reverse inequality comes from the fact that $e_1 + v$ belongs to the $\|\cdot\|$ -unit ball by definition.

■

Proof of Theorem 2.4. Property 2.5(f) immediately implies that $(l_2, \|\cdot\|)$ is not AUC, in fact that $\bar{\delta}_{\|\cdot\|}(t) = 0$ for $0 < t \leq 1$. On the other hand, $(l_2, \|\cdot\|)$ is AMUC with $\tilde{\delta}_{\|\cdot\|}(t) \geq \sqrt{1 + t^2/2} - 1$ for $0 < t \leq 1$. In fact, let $0 < t \leq 1$, and let $x = \sum_{i=1}^{\infty} x_i e_i$ with $\|x\| = 1$. We may suppose that $x_1 \geq 0$.

Let $\varepsilon > 0$, and let $N \geq 1$ be such that the vector $z = \sum_{i=1}^N x_i e_i$ has norm $\|z\| \geq 1 - \varepsilon$. Let $f = \sum_{i=1}^\infty b_i e_i^*$ be a $\|\cdot\|$ -norming functional for z , i.e. $\|f\|^* = 1$ and $f(z) = \|z\|$. We may assume $b_i = 0$ for $i > N$, therefore, $f = \sum_{i=1}^N b_i e_i^*$. By 2.5(e) and the fact that $x_1 \geq 0$ we obtain $b_1 \geq 0$.

Consider now the closed span Y_ε of the vectors $(e_i)_{i=N+1}^\infty$, and let $y \in Y_\varepsilon$ with $\|y\| = t$. We deduce from 2.5(d) that $\|y\| = \|y\|_2$. Write $y = y^+ - y^-$ where y^+ and y^- have non-negative coordinates. Then $\max(\|y^+\|, \|y^-\|) \geq t/\sqrt{2}$.

We consider two cases:

Case 1. Suppose that $s := \|y^+\| \geq t/\sqrt{2}$. Consider

$$g := \frac{1}{\sqrt{1+s^2}}(f - y^+).$$

Then

$$\|g\|_2^2 = \frac{\|f\|_2^2 + \|y^+\|_2^2}{1+s^2} \leq \frac{1+s^2}{1+s^2} = 1,$$

since $\|f\|_2 \leq \|f\|^* = 1$ and $\|y^+\|_2 = \|y^+\| = s$. Moreover,

$$\frac{|b_1 - (y^+)_i|}{\sqrt{1+s^2}} \leq \frac{\max(|b_1|, \|y^+\|_2)}{\sqrt{1+s^2}} \leq \frac{1}{\sqrt{1+s^2}} \leq 1$$

for all $i \geq N+1$. Here we use the fact that $b_1 \geq 0$, $y^+ \geq 0$ and $t \leq 1$. Note that $0 \leq (y^+)_i \leq t$. We also have $|b_1 + b_i| \leq \|f\|^* \leq 1$ for all $2 \leq i \leq N$. Thus, $\|g\|^* \leq 1$. Clearly,

$$g(x - y) = \frac{f(x) + \|y^+\|_2^2}{\sqrt{1+s^2}} = \frac{f(z) + \|y^+\|_2^2}{\sqrt{1+s^2}} = \frac{\|z\| + s^2}{\sqrt{1+s^2}}.$$

Since $\|z\| \geq 1 - \varepsilon$, we get that $g(x - y) \geq \sqrt{1+s^2} - \varepsilon \geq \sqrt{1+t^2/2} - \varepsilon$. Therefore, $\|x - y\| \geq \sqrt{1+t^2/2} - \varepsilon$.

Case 2. Let now $\|y^-\| \geq t/\sqrt{2}$ and consider

$$h = \frac{f - y^-}{\sqrt{1+\|y^-\|_2^2}}.$$

Similarly to the above reasoning we can show that $\|h\|^* \leq 1$ and

$$h(x + y) = \frac{f(z) + \|y^-\|_2^2}{\sqrt{1+\|y^-\|_2^2}} \geq \sqrt{1+t^2/2} - \varepsilon.$$

Combining both cases we obtain that

$$\inf_{y \in S_{Y_\varepsilon}} \max(\|x + ty\|, \|x - ty\|) \geq \sqrt{1 + t^2/2} - \varepsilon,$$

and consequently,

$$\sup_{\text{codim}(Y) < \infty} \inf_{y \in S_Y} \max(\|x + ty\|, \|x - ty\|) \geq \sqrt{1 + t^2/2} - \varepsilon,$$

where Y denotes a closed linear subspace of $(l_2, \|\cdot\|)$.

Letting $\varepsilon \rightarrow 0$, and then letting x run over all vectors in $S_{(l_2, \|\cdot\|)}$, we have $\tilde{\delta}_{\|\cdot\|}(t) \geq \sqrt{1 + t^2/2} - 1$ for all $0 < t \leq 1$, and thus, $(l_2, \|\cdot\|)$ is AMUC. ■

3. A renorming of l_2 which is midpoint locally uniformly convex does not have the Kadec-Klee property

Anderson [A] asked the question of whether reflexive midpoint locally uniformly convex Banach spaces have the Kadec-Klee property. P. Dowling, Zh. Hu and M. Smith [DHS] gave a negative answer to Anderson's question. Here we give another (simpler) example of a MLUC space which does not have KK-property. We will define a new equivalent norm in l_2 using the renorming $\|\cdot\|$ of l_2 defined in the previous section. To this end for an element $x = \sum_{n=1}^{\infty} a_n e_n$, denote

$$|x| = \left(\sum_{n=1}^{\infty} 2^{-n} a_n^2 \right)^{1/2}.$$

Consider now $(l_2, |||\cdot|||)$, where

$$||| \sum_{n=1}^{\infty} a_n e_n ||| := \left(\left\| \sum_{n=1}^{\infty} a_n e_n \right\|^2 + |x|^2 \right)^{1/2}.$$

$|||\cdot|||$ is an equivalent norm in l_2 . Moreover, it follows from $\|e_1\| = \|e_1 + e_n\|$ for all $n \geq 2$ (property 2.5(f) from the previous section) that $(e_1 + e_n)/\|e_1 + e_n\|$ tends weakly to $e_1/\|e_1\|$ and thus $(l_2, |||\cdot|||)$ does not have KK-property.

It remains to prove that $|||\cdot|||$ is MLUC. To see this, suppose that $1 = |||x||| = \lim_{n \rightarrow \infty} |||x \pm x_n|||$, $x_n \in X$. Thus,

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \left(|||x + x_n|||^2 + |||x - x_n|||^2 \right) / 2 \\ &\geq \|x\|^2 + \limsup_{n \rightarrow \infty} \left(|x + x_n|^2 + |x - x_n|^2 \right) / 2 \\ &= \|x\|^2 + |x|^2 + \limsup_{n \rightarrow \infty} |x_n|^2 = 1 + \limsup_{n \rightarrow \infty} |x_n|^2. \end{aligned}$$

Thus, $|x_n| \rightarrow 0$ and therefore, $x_n \rightarrow 0$ weakly. Using this we have the following chain of inequalities and equalities

$$\begin{aligned}
1 &\geq \frac{1}{2} \limsup_{n \rightarrow \infty} \left(\|x + x_n\|^2 + \|x - x_n\|^2 \right) + |x|^2 \\
&\geq \frac{1}{2} \left(\|x\|^2 + \limsup_{n \rightarrow \infty} (\max(\|x + x_n\|^2, \|x - x_n\|^2)) \right) + |x|^2 \\
&\geq \frac{1}{2} \left(\|x\|^2 + \|x\|^2 \left(1 + \tilde{\delta}_{\|\cdot\|} \left(\frac{\limsup_{n \rightarrow \infty} \|x_n\|}{\|x\|} \right) \right)^2 \right) + |x|^2 \\
&\geq \|x\|^2 + |x|^2 + \|x\|^2 \tilde{\delta}_{\|\cdot\|} \left(\frac{\limsup_{n \rightarrow \infty} \|x_n\|}{\|x\|} \right) \\
&= 1 + \|x\|^2 \tilde{\delta}_{\|\cdot\|} \left(\frac{\limsup_{n \rightarrow \infty} \|x_n\|}{\|x\|} \right).
\end{aligned}$$

Thus, $\limsup_{n \rightarrow \infty} \|x_n\| = 0$, whence $\limsup_{n \rightarrow \infty} |||x_n||| = 0$, and therefore, the norm $||| \cdot |||$ is MLUC. ■

4. A relation between the moduli of asymptotic uniform smoothness and sign-asymptotic uniform smoothness

Let $(X, \|\cdot\|)$ be a Banach space. For $x \in S_X$ and $t > 0$ define

$$\bar{\rho}(x, t) := \inf_{\text{codim}(Y) < \infty} \sup_{y \in S_Y} (\|x + ty\| - 1),$$

where Y denotes a closed subspace of X .

Recall that the *modulus of asymptotic uniform smoothness* is

$$\bar{\rho}_X(t) = \sup_{x \in S_X} \bar{\rho}(x, t),$$

and the norm of X is called *asymptotically uniformly smooth* (in short AUS) if $\lim_{t \rightarrow 0} \bar{\rho}_X(t)/t = 0$.

Similarly to the case of AMUC property we may weaken the above notion by introducing a sign, namely we can define

$$\tilde{\rho}(x, t) := \inf_{\text{codim}(Y) < \infty} \sup_{y \in S_Y} [\min(\|x + ty\|, \|x - ty\|) - 1],$$

where Y denotes again a closed subspace of X and

$$\tilde{\rho}_X(t) = \sup_{x \in S_X} \tilde{\rho}(x, t),$$

which we shall call *modulus of sign-asymptotic uniform smoothness*. Finally, we shall temporarily call the norm in X *sign-asymptotically uniformly smooth* (SAUS) if $\lim_{t \rightarrow 0} \tilde{\rho}_X(t)/t = 0$.

In fact, although formally the property SAUS is a weakening of AUS, we will see that SAUS coincides with AUS. To this end, observe first that in any Banach space X , $\tilde{\rho}(x, t) \leq \bar{\rho}(x, t) \leq t$, $t > 0$, for all $x \in S_X$. We have the following weaker reverse inequality.

Proposition 4.1. *Let $(X, \|\cdot\|)$ be a Banach space. For all $x \in S_X$ and $t > 0$, we have*

$$\tilde{\rho}\left(x, \frac{3t^2}{\bar{\rho}(x, t)}\right) \geq \bar{\rho}(x, t).$$

Proof. Fix $x \in S_X$ and $t > 0$. We may assume $\bar{\rho}(x, t) > 0$ since otherwise the result is trivial. Let $0 < \alpha < 1$ and let Y be a closed subspace of X of finite codimension. Choose $y \in S_Y$ such that

$$\|x + ty\| > 1 + \alpha\bar{\rho}(x, t).$$

Let further, $Z := Y \cap \ker(f)$, where $f \in X^*$ is a norming functional for $x + ty$. Therefore, we have

$$(4.1) \quad \|x + ty + z\| \geq \|x + ty\| \quad \text{for every } z \in Z.$$

Choose now $z \in S_Z$ such that

$$\|x + tz\| > 1 + \alpha\bar{\rho}(x, t).$$

By convexity, for all $\lambda \geq 1$ we will have

$$\|x + \lambda tz\| > 1 + \lambda\alpha\bar{\rho}(x, t).$$

In particular, for $\lambda = 2t/(\alpha\bar{\rho}(x, t))$ we obtain

$$\left\|x + \frac{2t}{\alpha\bar{\rho}(x, t)}tz\right\| > 1 + 2t.$$

Hence,

$$\left\|x - ty + \frac{2t}{\alpha\bar{\rho}(x, t)}tz\right\| > 1 + 2t - t = 1 + t > 1 + \alpha\bar{\rho}(x, t).$$

On the other hand, by (4.1) and the choice of y ,

$$\left\| x + ty - \frac{2t}{\alpha\bar{\rho}(x,t)}tz \right\| \geq \|x + ty\| > 1 + \alpha\bar{\rho}(x,t).$$

Let

$$w := y - \frac{2tz}{\alpha\bar{\rho}(x,t)}.$$

The above inequalities entail $\min(\|x + tw\|, \|x - tw\|) > 1 + \alpha\bar{\rho}(x,t)$. Note that

$$\frac{3t}{\alpha\bar{\rho}(x,t)} \geq \frac{2t}{\alpha\bar{\rho}(x,t)} + 1 \geq \|w\| \geq \frac{2t}{\alpha\bar{\rho}(x,t)} - 1 \geq 1.$$

Set $w' := w/\|w\|$. Then, $w' \in S_Y$ and by convexity

$$\min\left(\left\|x + \frac{3t^2}{\alpha\bar{\rho}(x,t)}w'\right\|, \left\|x - \frac{3t^2}{\alpha\bar{\rho}(x,t)}w'\right\|\right) > 1 + \alpha\bar{\rho}(x,t).$$

Therefore,

$$\tilde{\rho}\left(x, \frac{3t^2}{\alpha\bar{\rho}(x,t)}\right) \geq \alpha\bar{\rho}(x,t).$$

By continuity of $\tilde{\rho}(x, \cdot)$, letting $\alpha \uparrow 1$, we obtain

$$\tilde{\rho}\left(x, \frac{3t^2}{\bar{\rho}(x,t)}\right) \geq \bar{\rho}(x,t),$$

which completes the proof of the proposition. ■

We obtain the following consequence.

Corollary 4.2. *Let $(X, \|\cdot\|)$ be a Banach space. Then for all $t > 0$,*

$$\tilde{\rho}_X\left(\frac{3t^2}{\bar{\rho}_X(t)}\right) \geq \bar{\rho}_X(t).$$

Proof. Let $t > 0$ and choose $(x_n)_{n=1}^\infty \subset S_X$ so that $\bar{\rho}(x_n, t) \rightarrow \bar{\rho}_X(t)$. Then, using Proposition 4.1 we have

$$\liminf_{n \rightarrow \infty} \tilde{\rho}\left(x_n, \frac{3t^2}{\bar{\rho}_X(t)}\right) = \liminf_{n \rightarrow \infty} \tilde{\rho}\left(x_n, \frac{3t^2}{\bar{\rho}(x_n, t)}\right) \geq \liminf_{n \rightarrow \infty} \bar{\rho}(x_n, t) = \bar{\rho}_X(t).$$

Therefore,

$$\tilde{\rho}_X\left(\frac{3t^2}{\bar{\rho}_X(t)}\right) = \sup_{x \in S_X} \tilde{\rho}\left(x, \frac{3t^2}{\bar{\rho}_X(t)}\right) \geq \bar{\rho}_X(t).$$

■

Now we are ready to prove the equivalence between the properties AUS and SAUS.

Corollary 4.3. *The norm of a Banach space X is SAUS if and only if it is AUS.*

Proof. Clearly the property AUS implies SAUS because $\bar{\rho}_X(t) \geq \tilde{\rho}_X(t)$ for every $t > 0$. Assume now that the space X is not AUS. Then there is an $\varepsilon > 0$ and $t_n \downarrow 0$ so that $\bar{\rho}_X(t_n) > \varepsilon t_n$ for every $n \geq 1$. Hence,

$$\tilde{\rho}_X\left(\frac{3t_n}{\varepsilon}\right) = \tilde{\rho}_X\left(\frac{3t_n^2}{\varepsilon t_n}\right) \geq \tilde{\rho}_X\left(\frac{3t_n^2}{\bar{\rho}_X(t_n)}\right) \geq \bar{\rho}_X(t_n) > \varepsilon t_n = \frac{\varepsilon^2}{3}\left(\frac{3t_n}{\varepsilon}\right),$$

and since $\lim_{n \rightarrow \infty} (3t_n)/\varepsilon = 0$ we conclude that X is not SAUS. ■

5. AMUC under renorming

With regard to the properties of asymptotic uniform convexity (AUC) and asymptotic midpoint uniform convexity (AMUC) the following question is natural:

Question: Are the properties AMUC and AUC isomorphically equivalent?

We give a partial positive answer to this question.

Proposition 5.1. *Let $(X, \|\cdot\|)$ be a Banach space with an unconditional Schauder basis with unconditional basis constant K_u . If $|||\cdot|||$ denotes the standard equivalent norm that makes the unconditional basis 1-unconditional, then for every $t > 0$ one has $\bar{\delta}_{|||\cdot|||}(t) \geq \tilde{\delta}_{\|\cdot\|}\left(\frac{t}{K_u}\right)$. In particular, if the norm $\|\cdot\|$ is AMUC then the norm $|||\cdot|||$ is AUC.*

Proof. Denote $(e_n)_{n=1}^\infty$ the unconditional basis for X . For $x = \sum_{n=1}^\infty a_n e_n \in X$ we have

$$|||x||| = \sup_{\sigma_n = \pm 1} \left\| \sum_{n=1}^\infty \sigma_n a_n e_n \right\|.$$

We have $\|x\| \leq |||x||| \leq K_u \|x\|$ for all $x \in X$.

Fix now $t > 0$ and let $x = \sum_{i=1}^\infty a_i e_i \in X$ with $|||x||| = 1$. Let $\varepsilon > 0$ and let $N \geq 1$ be such that the vector $z = \sum_{i=1}^N a_i e_i$ satisfies $|||x - z||| \leq \varepsilon$. We can find signs $(\sigma_i)_{i=1}^N$ such that the element $z' = \sum_{i=1}^N \sigma_i a_i e_i$ satisfies $\|z'\| = |||z|||$. Since

the norm $\|\cdot\|$ is AMUC, we may choose a subspace W of finite codimension such that for every $y \in W$ with $\|y\| = \|z'\|$, we have

$$\max \left(\left\| z' + \frac{ty}{K_u} \right\|, \left\| z' - \frac{ty}{K_u} \right\| \right) \geq \|z'\| \left[1 + \tilde{\delta}_{\|\cdot\|} \left(\frac{t}{K_u} \right) - \varepsilon \right].$$

Note that, by convexity, the above holds true also for every $y \in W$ with $\|y\| \geq \|z'\|$. Consider any $y \in W$ such that $\|y\| = 1$ and $\text{supp}(y) \subset [N+1, \infty)$. Then because $1 \geq \|z\| = \|z'\|$, we have $\|y\| \geq \frac{\|z'\|}{K_u}$, and thus

$$\max (\|z' + ty\|, \|z' - ty\|) \geq \|z'\| \left[1 + \tilde{\delta}_{\|\cdot\|} \left(\frac{t}{K_u} \right) - \varepsilon \right].$$

Since $\|z + ty\| \geq \max (\|z' + ty\|, \|z' - ty\|)$, we obtain that

$$\|z + ty\| \geq \|z\| \left[1 + \tilde{\delta}_{\|\cdot\|} \left(\frac{t}{K_u} \right) - \varepsilon \right].$$

We then have

$$\|x + ty\| \geq \|z + ty\| - \|x - z\| \geq (1 - \varepsilon) \left[1 + \tilde{\delta}_{\|\cdot\|} \left(\frac{t}{K_u} \right) - \varepsilon \right] - \varepsilon.$$

This gives

$$\inf \|x + ty\| \geq (1 - \varepsilon) \left[1 + \tilde{\delta}_{\|\cdot\|} \left(\frac{t}{K_u} \right) - \varepsilon \right] - \varepsilon,$$

where the infimum is taken over all $y \in W$ with $\text{supp}(y) \subset [N+1, \infty)$ and with $\|y\| = 1$. Consequently,

$$\sup_{\text{codim}(Y) < \infty} \inf_{y \in S_{(Y, \|\cdot\|)}} \|x + ty\| \geq (1 - \varepsilon) \left[1 + \tilde{\delta}_{\|\cdot\|} \left(\frac{t}{K_u} \right) - \varepsilon \right] - \varepsilon,$$

where Y denotes a closed linear subspace of X .

Letting $\varepsilon \rightarrow 0$, and then taking infimum over all $x \in S_{(X, \|\cdot\|)}$, we get $\bar{\delta}_{\|\cdot\|}(t) \geq \tilde{\delta}_{\|\cdot\|} \left(\frac{t}{K_u} \right)$. ■

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