

Catalan’s constant is irrational

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September 24, 2026

Abstract

We prove that Catalan’s constant $G = \sum_{j \geq 0} (-1)^j / (2j + 1)^2$ is irrational. The proof builds explicit determinants whose entries are rational linear combinations of 1, G and $\zeta(2)$, arranges a Taylor-contact cancellation that removes the $\zeta(2)$ terms, and then compares a sharp lower bound on the prime valuations of these determinants with a sharp upper bound on their real size. The two bounds are incompatible at the scale of the square of the matrix size, so G cannot be rational. The argument is self-contained apart from the ordinary prime number theorem; every finite numerical statement is certified by exact rational arithmetic.

1 Introduction

1.1 What we prove

Catalan’s constant is the real number

$$G = \beta(2) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \cdots \approx 0.9159655941772190 \dots$$

where β is the Dirichlet beta function. Equivalently $G = L(2, \chi_{-4})$ for the odd quadratic character of conductor 4. The same number is one quarter of the volume of the smallest orientable complete finite-volume hyperbolic three-manifold with two cusps, and it appears in lattice sums and in a large family of integrals. Its arithmetic type is the question we settle.

Theorem 1.1. *Catalan’s constant is irrational.*

The word “irrational” means that $G \neq p/q$ for all integers p, q . Everything below is aimed at Theorem 1.1; every other result in the paper is a lemma, proposition or estimate feeding it.

1.2 Why a good approximation is not enough

Because we can compute G to billions of digits, one might think its arithmetic is within reach. This is false, and the reason drives the whole proof.

Suppose G really is a fraction p/q . Then for any integers A, B the combination

$$A G - B = \frac{A p - q B}{q}$$

is a rational with denominator dividing q , so it is either exactly zero or at least $1/q$ in absolute value: it cannot be “small but nonzero”. This gives the standard criterion.

Proposition 1.2 (Rationality criterion). *A real number α is irrational if and only if there are integers $A_m \neq 0$ and B_m with*

$$0 < |A_m \alpha - B_m| \longrightarrow 0 \quad (m \rightarrow \infty).$$

Proof. Take $\alpha = p/q$ rational. If $A_m \alpha - B_m \neq 0$ then $|A_m \alpha - B_m| \geq 1/q$, independent of m , so the limit cannot be 0. Conversely, if α is irrational, the continued fraction of α produces convergents P_m/Q_m with $|Q_m \alpha - P_m| < 1/Q_m \rightarrow 0$, and these forms are nonzero (an exact equality would make $\alpha = P_m/Q_m$ rational). $\square$

So to prove irrationality we must build nonzero integer combinations that tend to zero. The hard part is the word *nonzero*: approximations P_m/Q_m only give $|Q_m G - P_m| = Q_m |G - P_m/Q_m|$, and multiplying by the denominator Q_m can easily destroy the decay.

1.3 The obstruction for Catalan’s constant, and earlier work

For odd positive values $\beta(1), \beta(3), \dots$ the question is trivial, because these are rational multiples of powers of π . The even values are genuinely arithmetic. Rivoal and Zudilin proved that infinitely many even beta values are irrational, and that at least one of $\beta(2), \beta(4), \dots, \beta(14)$ is irrational [19, Theorems 1 and 2]. Zudilin reduced this list to the six values through $\beta(12)$ [28, Theorem 1], and Lai and Zhou to the five values through $\beta(10)$ [15, Theorem 6.1]. Fischler obtained stronger quantitative linear-independence results for families of Dirichlet L -values [9, Theorems 1 and 2]. All of these are “at least one of a family” statements; none identifies $\beta(2)$ itself.

For a single value one wants explicit integer forms $A_m G - B_m \rightarrow 0$. Apéry’s recurrences for $\zeta(2)$ and $\zeta(3)$ [2] and Beukers’s integral proofs [3] succeed because the analytic decay survives the necessary denominator clearing. Zudilin constructed Apéry-like recurrences and integral representations for G [26, Theorems 1–3], and the discussion after his Theorem 1 makes the problem explicit: the rational quotients converge, but after clearing denominators the integer forms fail to tend to zero. The same obstruction appears in the Padé approximation work of Rivoal [18, Theorems 1 and 2] and Krattenthaler–Rivoal [13, Theorems 1 and 2], in the hypergeometric identity work of Krattenthaler and Zudilin [14, Section 2 and Theorem 2], in Nesterenko’s half-integer hypergeometric approximations [16], and in the Rhin–Viola exponent $0.6293\dots$ reported by Viola [23]. Explicit families with exponents below 1, such as those of Eskandari [8, Theorem 1.1], still fall short of Proposition 1.2: an exponent below 1 does not by itself make $A_m G - B_m \rightarrow 0$.

At neighboring characters, Calegari, Dimitrov and Tang proved the $\mathbb{Q}$ -linear independence of $1, \pi^2, L(2, \chi_{-3})$ [5, Theorem A], and their Remark 11.1.17 again records the denominator cost as the obstruction for two Catalan families. Calegari proved irrationality of a 2-adic analogue of G [4, Theorem 4.2]; as Calegari, Dimitrov and Tang explain [6, Section 5.2, footnote 2], the real identity carries an extra π^2 term that the 2-adic identity does not, which is exactly why the 2-adic result does not transfer. In our construction the cancellation of an extra $\zeta(2)$ term is likewise essential. Sun has also announced the same qualitative statement [22, Theorem 1.1]; nothing from that preprint is used here.

1.4 The plan: one determinant, two incompatible estimates

Instead of a single linear form we use a whole integer matrix. Its determinant Δ_N is signed, but that is fine: we only need that Δ_N is a nonzero rational number whose size we can bound from below and from above.

Step 1 (build and cancel). For each N we build an $n \times n$ determinant Δ_N with $n = 48N$. Its entries are moments of two elementary kernels against polynomial rows. Each moment is a rational

combination of 1, G and $\zeta(2)$. The rows are chosen so that the coefficients of $\zeta(2)$ cancel *exactly*, by matching Taylor series. Thus, under the single hypothesis $G \in \mathbb{Q}$, every entry and hence Δ_N lies in $\mathbb{Q}$. This is Section 2.

Step 2 (it is nonzero). For a large prime p take $N = p$ and prove $\Delta_p \neq 0$, indeed $v_p(\Delta_p) = -96p$ exactly. The proof reduces the $48p \times 48p$ matrix modulo p to three fixed rational 48×48 matrices, and an exact computation over $\mathbb{F}_{101}$ proves all three invertible. This is Section 4.

Step 3 (a lower bound from valuations). A nonzero rational number is pinned down by its prime valuations, $\log |\Delta_N| = \sum_p v_p(\Delta_N) \log p$. Careful p -adic estimates of the moments bound how negative $v_p(\Delta_N)$ can be at each prime, and summing $\log p$ against the prime number theorem gives a lower bound. This is Section 3, with the prime 2 handled in Section 2.

Step 4 (an upper bound from an integral). Independently of any arithmetic hypothesis, Δ_N has an integral representation as a product of Vandermonde determinants. Estimating that integral gives an upper bound on $|\Delta_N|$. This is Sections 5 and 6, with the numerical constants certified in Section 7.

The payoff is a pair of bounds of opposite type on the same normalized logarithm. Define

$$\mathcal{L}_N = \frac{\log |\Delta_N|}{(48N)^2} - \frac{1}{2} \log 2. \quad (1)$$

If $G \in \mathbb{Q}$, then along any sequence of N with $\Delta_N \neq 0$,

$$\liminf_{N \rightarrow \infty} \mathcal{L}_N > -2.29084 \quad (\text{Proposition 3.4}),$$

and unconditionally

$$\limsup_{N \rightarrow \infty} \mathcal{L}_N \leq -2.290939875 < -2.2909 \quad (\text{Proposition 7.1}).$$

If G were rational we could take the sequence $N = p$, $p \rightarrow \infty$ prime, on which both bounds apply and $\Delta_p \neq 0$. Then $-2.29084 < \liminf \mathcal{L}_p \leq \limsup \mathcal{L}_p \leq -2.290939875$, but $-2.29084 > -2.2909$. Contradiction. Hence G is irrational.

The whole proof is therefore an exercise in making the two constants 2.29084 and 2.290939875 genuinely correct, with a margin.

Zudilin's determinantal criterion [27] is a useful precedent, but its determinant is a positive Hankel determinant, whereas ours is mixed and signed, so nonvanishing and the real estimate need separate arguments; the logarithmic interactions of the resulting Vandermonde products are estimated as in logarithmic potential theory [20].

1.5 What makes the comparison close enough

Three design choices are responsible for the narrow positive gap.

Integral rows with two layers of divisibility. The rows are integral Chebyshev polynomials in disguise, which supplies both the Taylor contact that cancels $\zeta(2)$ and useful arithmetic divisibility. At the large odd primes that dominate the leading bound, the entries have denominators of order p^2 and of order p ; both layers must be tracked separately, because cancelling only the first layer would leave a denominator bound too weak to win.

Nonvanishing by arithmetic, not by positivity. The determinant is mixed and signed, so there is no positivity to fall back on. Instead we reduce a growing matrix to a fixed finite one using Frobenius and two polynomial bases, and check the fixed matrices by exact modular arithmetic; here the certificate over $\mathbb{F}_{101}$ enters.

Respect for both branches of the parametrization. The rows live on two branches of a rational parametrization, and the real estimate treats both. A bounded holomorphic interpolant controls the mixed evaluations when the integration nodes are spread out; a Hadamard bound controls them when the nodes bunch up. The interpolation is uniform even as nodes collide, because the evaluation determinant cancels algebraically inside a finite-dimensional Hardy space, so no inverse-Vandermonde estimate is needed. The resulting Vandermonde products turn the many-variable supremum into an explicit energy problem, whose answer is the rational trial data of Section 7.

1.6 How to read this paper

The logical spine is short: Section 2 builds Δ_N and proves it rational under $G \in \mathbb{Q}$; Section 3 and the prime-2 part of Section 2 give the lower bound; Section 4 supplies the nonzero determinants; Sections 5 and 6 give the upper bound; Section 7 certifies the finite constants; and Section 8 assembles the contradiction. A reader who wants only the shape of the argument can read this introduction and Section 8; the intermediate sections are the estimates that make the two constants correct.

The only outside analytic input in the arithmetic half is the ordinary prime number theorem, used in the form $\theta(y) = \sum_{p \leq y} \log p \sim y$ and proved locally in the form actually needed [25, Step VI, p. 707]. The real half uses Andréief's integration identity [10, Equation (1.7) and Section 2.2], Cauchy's double alternant [12, Section 2.1, Equation (2.7)], the classical bounded-interpolation picture [21] and a Chebyshev expansion of the logarithmic kernel [11, Lemma 3.1], all restated and proved where used. Section 8 also records the hyperbolic-volume consequences.

1.7 Conventions

All logarithms are natural. We use $v_p(p) = 1$ and $v_p(0) = +\infty$; for real estimates we put $\log 0 = -\infty$. Constants in asymptotic estimates may depend on the fixed rational value hypothetically assigned to G , but never on N or on the integration variables. Finite certificates are specified by rational data and rational arithmetic, so they can be checked without numerical logarithms or root finding.

2 Polynomial rows and mixed moments

This section builds the determinant Δ_N and proves the one arithmetic fact we need from it: under the hypothesis $G \in \mathbb{Q}$, every entry of Δ_N is rational. The cancellation of $\zeta(2)$ is the whole reason to design the rows as we do, so we motivate it first and then carry out the construction in full.

2.1 The two kernels and why moments are rational

Two elementary kernels appear: on $(-1, 1)$ the weight $|t| dt/f(t)$, and on $(0, 1)$ the weight ds . Both integrate against $1/(1-ts)$, and the geometric series $1/(1-ts) = \sum_{j \geq 0} (ts)^j$ converts these moments into simple sums. The moments are governed by exactly two classical constants, G and $\zeta(2)$, and this is what makes the construction arithmetic.

For each positive integer N set

$$\begin{aligned} n &= 48N, & a &= 11N, & b &= 7N, & q &= g = 4N, & h &= 2N, \\ L &= n + a = n + b + q = 59N, & C &= L + g = 63N, \\ H &= C + h = 65N, & A &= a + 2g = 19N. \end{aligned} \tag{2}$$

The matrix has size $n = 48N$; the integers a, b, q, g, h and the resulting L, C, H, A are the bookkeeping that makes both the arithmetic and the real estimates come out. The numbers 48, 11, 7, 4, 2 are not natural, but the outcome of optimizing the two final constants.

Write $f(t) = \sqrt{1-t^2}$ with the positive branch on $(-1, 1)$, and let $w = t/(1+f)$. Then $t = 2w/(1+w^2)$ and $f = (1-w^2)/(1+w^2)$. Let the star involution send w to w^{-1} ; it fixes t and sends f to $-f$. For $0 \leq r < n$ define the row polynomials

$$R_r = (1-t)^h t^{C-1} w^{r-g}, \quad P_r = \frac{R_r + R_r^*}{2}, \quad D_r = \frac{t(R_r^* - R_r)}{2f}. \quad (3)$$

Near $t = 0$ we read f through its Taylor branch with $f(0) = 1$, so these are polynomial identities rather than statements about irrationalities.

Lemma 2.1. *Let T_d, U_d be the Chebyshev polynomials, normalized by $T_0 = 1$, $T_1 = x$, $U_{-1} = 0$, $U_0 = 1$ and $S_{d+1} = 2xS_d - S_{d-1}$. With $d = |r - g|$,*

$$P_r = (1-t)^h t^{C-1} T_d(1/t), \quad D_r = \operatorname{sgn}(r-g)(1-t)^h t^{C-1} U_{d-1}(1/t),$$

and these are integer polynomials, the second being zero when $r = g$. Moreover

$$\operatorname{supp} P_r \subseteq \{A, \dots, H-1\}, \quad \operatorname{supp} D_r \subseteq \{A+1, \dots, H-1\}. \quad (4)$$

Proof. The identities $(w + w^{-1})/2 = 1/t$ and $(w^{-1} - w)/2 = f/t$ turn $R_r \pm R_r^*$ into the displayed Chebyshev expressions; these are the standard Laurent formulas [NIST DLMF, Equations (18.5.1)–(18.5.2)]. The degree bound $d \leq n - 1 - g$ gives the stated supports after multiplying by $(1-t)^h t^{C-1}$. $\square$

The key algebraic fact is the Taylor contact

$$\frac{tP_r}{f} - D_r = \frac{tR_r}{f} = O(t^{C+r-g}) = O(t^L). \quad (5)$$

Since $w = t/2 + O(t^3)$, the combination tR_r/f vanishes to order at least L , exactly the order of the highest column index j . This is the cancellation mechanism, and it removes the $\zeta(2)$ coefficient below.

2.2 The moments and the determinant

For nonnegative integers i, j define the two mixed moments

$$M(i, j) = \int_{-1}^1 \int_0^1 \frac{|t|}{f(t)} \frac{t^i s^j}{1-ts} ds dt, \quad Z(i, j) = \int_0^1 \int_0^1 \frac{t^i s^j}{1-ts} ds dt, \quad (6)$$

extended bilinearly to polynomial first arguments. Both integrals converge absolutely: integrating in s leaves $(-\log(1-u))/\sqrt{1-u^2}$, which is integrable on $(0, 1)$, and bounded polynomial factors do not disturb this. Hence every determinant expansion below is absolutely integrable and Fubini applies.

For a raw column index $b \leq j < L$ put

$$F_j(r) = M(P_r, j) - \frac{3}{2}Z(D_r, j) \quad (0 \leq r < n).$$

The determinant used throughout the proof is

$$\Delta_N = \det_{0 \leq r, k < n} \left(M(P_r, s^{b+k}(1-s)^q) - \frac{3}{2} Z(D_r, s^{b+k}(1-s)^q) \right). \quad (7)$$

If $\mathcal{F}$ is the $n \times (n+q)$ matrix of raw columns F_j , then $\Delta_N = \det(\mathcal{FT})$ where the integer filter matrix $\mathcal{T}$ has entries

$$\mathcal{T}_{j,k} = (-1)^{j-b-k} \binom{q}{j-b-k}, \quad b \leq j < L, \quad 0 \leq k < n. \quad (8)$$

This filter is a fourth finite difference in disguise; it is why the raw column range is $j < L$ and why the arithmetic estimate can be layered (Section 3).

2.3 Moment evaluation and cancellation of $\zeta(2)$

Set

$$c_l = 4^{-l} \binom{2l}{l}, \quad m_i = \int_{-1}^1 t^i \frac{|t|}{f(t)} dt = \begin{cases} \frac{2}{(i+1)c_{i/2}}, & i \text{ even,} \\ 0, & i \text{ odd,} \end{cases} \quad (9)$$

with $m_{-1} = 0$. The moment formula follows from symmetry, $m_0 = 2$, and integration by parts $(i+1)m_i = im_{i-2}$. Termwise integration gives

$$M(i, j) = \sum_{u \geq 0} \frac{m_{i+u}}{j+u+1}, \quad M(i, j) - M(i+1, j+1) = \frac{m_i}{j+1}. \quad (10)$$

Put $K_d^- = M(d, 0)$ and $K_d^+ = M(0, d)$. Their boundary recurrences are

$$dK_d^- = (d-1)K_{d-2}^- + m_{d-2} + m_{d-1}, \quad K_1^- = 2, \quad (11)$$

$$(d-1)K_d^+ = (d-2)K_{d-2}^+ + \frac{2}{d-1} \quad (d \geq 2), \quad (12)$$

from the telescoping identity $\frac{dm_{d+u} - (d-1)m_{d+u-2}}{u+1} = m_{d+u-2} - m_{d+u}$ (valid termwise and summing since $m_v \rightarrow 0$). The two independent starts are

$$K_0^- = K_0^+ = 4G, \quad K_1^+ = \frac{\pi^2}{4} = \frac{3}{2}\zeta(2). \quad (13)$$

Indeed $K_0^- = \int_0^{\pi/2} \log \frac{1+\sin \theta}{1-\sin \theta} d\theta = -4 \int_0^{\pi/4} \log \tan v dv = 4G$, and $K_1^+ = \sum_{k \geq 1} 1/(2k^2 c_k)$ together with $(\arcsin x)^2 = \sum_{k \geq 1} x^{2k}/(2k^2 c_k)$ at $x = 1$ gives $\pi^2/4 = \frac{3}{2}\zeta(2)$.

Now define $M^0(i, j)$ by the same recurrences (10)–(12) but with the starts $K_0^- = K_0^+$ and K_1^+ replaced by zero. Concretely, let k_d^-, k_d^+ satisfy (11)–(12) with

$$k_0^- = k_0^+ = k_1^+ = 0, \quad k_1^- = 2. \quad (14)$$

Then

$$M^0(i, j) = \begin{cases} k_{i-j}^- - \sum_{k=1}^j \frac{m_{i-j+k-1}}{k}, & i \geq j, \\ k_{j-i}^+ - \sum_{k=j-i+1}^j \frac{m_{k-(j-i)-1}}{k}, & i < j, \end{cases} \quad (15)$$

with the algebraic convention $M^0(-1, j) = k_{j+1}^+$ (not an integral). With $B_i^{(d)} = \sum_{k=1}^i k^{-d}$ and $B_0^{(d)} = 0$, put

$$Z^0(i, j) = \begin{cases} -B_i^{(2)}, & i = j, \\ \frac{B_i^{(1)} - B_j^{(1)}}{i - j}, & i \neq j. \end{cases} \quad (16)$$

Then the two kernels separate cleanly:

$$\begin{aligned} M(i, j) &= M^0(i, j) + 4G c_{(i-j)/2} + \frac{3}{2} \zeta(2) c_{(j-i-1)/2}, \\ Z(i, j) &= Z^0(i, j) + [i = j] \zeta(2). \end{aligned} \quad (17)$$

Here $c_z = 0$ unless z is a nonnegative integer, and $[i = j]$ is one if $i = j$ and zero otherwise. The reason is that the homogeneous parts of the boundary recurrences propagate the starts by exactly these two c -kernels, while diagonal reduction preserves both index differences; the Z formula is partial fractions plus the diagonal term.

For the arithmetic estimate of Section 3 we record the explicit solution of the rational recurrences. Define

$$H_z^* = \begin{cases} c_{z/2}, & z \geq 0 \text{ even}, \\ \frac{1}{z c_{(z-1)/2}}, & z \geq 1 \text{ odd}, \end{cases} \quad \frac{H_{z+2}^*}{H_z^*} = \frac{z+1}{z+2}. \quad (18)$$

Then

$$\begin{aligned} k_u^- &= \begin{cases} H_u^* \sum_{\substack{1 \leq z \leq u \\ z \text{ even}}} \frac{2}{z^2 (H_z^*)^2}, & u \text{ even}, \\ H_u^* \sum_{\substack{1 \leq z \leq u \\ z \text{ odd}}} \frac{2}{z}, & u \text{ odd}, \end{cases} \\ k_{u+1}^+ &= H_u^* \sum_{\substack{1 \leq z \leq u \\ z \equiv u \pmod{2}}} \frac{2}{z^2 H_z^*} \quad (u \geq 0). \end{aligned} \quad (19)$$

Indeed, dividing each recurrence by the appropriate H^* and using the ratio in (18) turns the increments into $2/(z^2 (H_z^*)^2)$ and $2/z$ for the minus recurrence and $2/(z^2 H_z^*)$ for the plus recurrence, with the stated starts; this proves all the formulas, including the empty-sum cases.

Now we can see the cancellation.

Proposition 2.2. *Each raw entry $F_j(r)$ lies in $\mathbb{Q} + \mathbb{Q}G$. Consequently, if $G \in \mathbb{Q}$, then $\Delta_N \in \mathbb{Q}$ for every positive integer N .*

Proof. By (17) the coefficient of $\zeta(2)$ in $F_j(r)$ is

$$\frac{3}{2} \left(\sum_i [t^i] P_r c_{(j-i-1)/2} - [t^j] D_r \right) = \frac{3}{2} [t^j] \left(\frac{tP_r}{f} - D_r \right) = 0$$

because $j < L$ and the contact (5) gives $tP_r/f - D_r = O(t^L)$. Written out, only the G term and the rational arrays survive:

$$F_j(r) = \sum_i [t^i] P_r (M^0(i, j) + 4G c_{(i-j)/2}) - \frac{3}{2} \sum_i [t^i] D_r Z^0(i, j). \quad (20)$$

Both sums are finite and involve only rational data, so $F_j(r) \in \mathbb{Q} + \mathbb{Q}G$; if $G \in \mathbb{Q}$ the filter $\mathcal{T}$ is integer and $\Delta_N \in \mathbb{Q}$. $\square$

The inevitable $\zeta(2)$ in the moments is removed by choosing D_r to match the Taylor series of tP_r/f through order L .

2.4 The estimate at the prime two

Assume now $G \in \mathbb{Q}$ (also inside $\mathbb{Q}_2$) and bound the 2-adic valuation of Δ_N . This is the first half of the lower bound; the odd primes are Section 3. Constants that may depend on the fixed rational value of G carry a subscript G .

Proposition 2.3. *With $\delta = (q + g + h)/n = 5/24$, for every positive integer N ,*

$$v_2(\Delta_N) \geq -\left(\frac{\delta}{2} + \frac{\delta^2}{8}\right)n^2 - O_G(n \log(n+2)) = -\frac{505}{4608}n^2 - O_G(n \log(n+2)). \quad (21)$$

The same bound holds for every full minor of the raw column matrix.

Proof. Introduce the 2-adic smoothed moments

$$M_s(i, j) = \sum_{k \geq 0} \frac{m_{i+k}}{j+k+1} \quad \text{in } \mathbb{Q}_2. \quad (22)$$

For even u the factorial formula gives $v_2(m_u) = 1 + u - v_2\left(\frac{u}{2}\right) \geq u + 1 - \log_2(u+1)$, and odd moments vanish. Hence for $0 \leq i, j < H$ every term of (22) has valuation at least $i - 2\log_2 H - 1$, so the tail converges uniformly and $v_2(M_s(i, j)) \geq i - 2\log_2 H - 1$.

The diagonal and boundary recurrences hold for M_s as well, with tails tending to zero 2-adically, so there are exactly two homogeneous discrepancies against $M^0(i, j) + 4G c_{(i-j)/2}$:

$$e_1 = 4G - M_s(0, 0), \quad e_2 = -M_s(0, 1).$$

Both are fixed elements of $\mathbb{Q}_2$, independent of N . Fix K_G with $v_2(e_1), v_2(e_2) \geq -K_G$. Split each raw column as

$$\begin{aligned} F_j(r) &= S_j(r) + E_j(r) + B_j(r), \\ S_j(r) &= M_s(P_r, j), \\ E_j(r) &= e_1 \sum_i [t^i] P_r c_{(i-j)/2}, \\ B_j(r) &= \sum_i [t^i] D_r \left(e_2 [i = j] - \frac{3}{2} Z^0(i, j) \right). \end{aligned} \quad (23)$$

Here S_j is smoothed, E_j carries the e_1 kernel and B_j the e_2 kernel. Coefficient bounds (Chebyshev valuations $v_2([x^u]T_d) \geq u - 1$, $v_2([x^u]U_d) \geq u$, and $v_2(c_l) \geq -2l$) give $v_2(a_{u,j}) \geq u + j - H + 1 - K_G$ and $v_2(b_{u,j}) \geq -2L_2 - 1 - K_G$ with $L_2 = \lfloor \log_2 H \rfloor$.

Expand $\det(\mathcal{FT})$ by Cauchy–Binet and fix a full raw minor. A term with m columns of type E , l of type B and $n - m - l$ of type S has valuation at least

$$-O_G(n \log(n+2)) + \min_{m+l \leq n} \left\{ -(H-b)m + \frac{3}{2}m^2 + \frac{1}{2}l^2 + C(n-m-l) \right\}.$$

Minimizing over reals, using $H - b = n(1 + \delta)$ and $C \geq n$, the minimum is at $l = n - m$ and equals

$$\frac{n^2}{2} - (2 + \delta)nm + 2m^2 = 2\left(m - \frac{(2+\delta)n}{4}\right)^2 - \left(\frac{\delta}{2} + \frac{\delta^2}{8}\right)n^2.$$

Summing over minors proves (21); the same expansion gives the bound for each raw minor. $\square$

The number 505/4608 is added to the odd-prime loss in Section 3 to obtain the constant -2.29084 of Proposition 3.4.

3 Odd primes and the finite-place lower bound

Throughout this section assume $G \in \mathbb{Q}$, so the columns F_j of Section 2 are rational. For an odd prime p , reduction modulo p always means reduction of a member of $\mathbb{Z}_p$; every congruence below includes the assertion that both sides are locally integral.

The raw matrix has $n + q$ columns while the filtered determinant has size n . At the large primes treated first, each raw column has at most two powers of p in its denominator. We make invertible column changes over $\mathbb{Z}_p$ and bound how many columns can still have denominator p^2 or p . Such bounds control every full minor of the raw matrix, and Cauchy–Binet transfers them to the integer column filter. The first reduction identifies residues after multiplication by p^2 ; for $p > H/2$ a second reduction identifies the remaining residues after multiplication by p .

3.1 Digit reduction of the rational moments

We first prove the reductions needed at primes $2\sqrt{H} < p \leq H$. We may suppose p does not divide the denominator of G : as $N \rightarrow \infty$ every fixed denominator prime eventually lies below $2\sqrt{H}$. Put

$$E(t) = (1 - t^2)^{(p-1)/2} = \sum_{d=0}^{p-1} E_d t^d \quad \text{in } \mathbb{F}_p[t], \quad \epsilon = (-1)^{(p-1)/2},$$

and set $E_d = 0$ outside $0 \leq d < p$. Within this range,

$$E_d = \begin{cases} c_{d/2} \pmod{p}, & d \text{ even}, \\ 0, & d \text{ odd}, \end{cases} \quad E_{p-1-d} = \epsilon E_d,$$

the first from $(-1)^k \binom{(p-1)/2}{k} \equiv 4^{-k} \binom{2k}{k}$ and the second by reversing coefficients.

Lemma 3.1 (Digit reductions). *For $0 \leq i, j < H$ let*

$$\ell = j \bmod p, \quad d = (j - i - 1) \bmod p, \quad i' = \frac{i + 1 + d - \ell}{p} - 1, \quad j' = \left\lfloor \frac{j}{p} \right\rfloor,$$

with $0 \leq \ell, d < p$. Then $i' \geq -1$, and, using the convention $M^0(-1, j') = k_{j'+1}^+$,

$$\begin{aligned} p^2 M^0(i, j) &\equiv E_d M^0(i', j') \pmod{p}, \\ p^2 Z^0(i, j) &\equiv \begin{cases} Z^0(\lfloor i/p \rfloor, j'), & i \equiv j \pmod{p}, \\ 0, & i \not\equiv j \pmod{p}. \end{cases} \end{aligned} \quad (24)$$

Proof. We first give the parity and carry calculation for the factors H_z^* of (18). Write $z = Pp + r$, $0 \leq r < p$, $0 \leq z < H$. Since $H < p^2/4$, all factors at reduced indices P are p -adic units. If z is even,

$$\begin{cases} H_z^* \equiv H_{Pc_{r/2}}^* \pmod{p}, & r \text{ even}, \\ v_p(H_z^*) = 1, & r \text{ odd}. \end{cases} \quad (25)$$

When r is even, P is even and the base- p digits of $z/2$ are $P/2, r/2$, so $(1+x)^p = 1+x^p$ gives the first line including $4^{-z/2}$. When r is odd, P is odd and the low digit of $z/2$ is $(p+r)/2$; the factorial formula for $\binom{z}{z/2}$ has exactly one carry, $\lfloor z/p \rfloor - 2\lfloor (z/2)/p \rfloor = 1$, with no p^2 contribution. For odd z one has

$$\frac{1}{H_z^*} \in \mathbb{Z}_p, \quad pH_z^* \equiv \begin{cases} \epsilon c_{r/2} H_P^*, & r \text{ even}, \\ 0, & r \text{ odd}. \end{cases} \quad (26)$$

If r is odd, both z and $c_{(z-1)/2}$ are units; if r is even, P is odd and digit expansion gives $c_{(Pp-1)/2} \equiv \epsilon c_{(P-1)/2}$ and $pH_{Pp}^* \equiv \epsilon H_P^*$. The recurrence $H_{z+2}^*/H_z^* = (z+1)/(z+2)$ through $r = 0, 2, \dots, p-1$ proves the nonzero line. Since $m_{z-1} = 2H_z^*$ for odd z and vanishes for even z ,

$$pm_{z-1} \equiv E_{p-1-r}m_{P-1} \pmod{p},$$

including $z = 0$ with $m_{-1} = 0$.

For $u = Pp + r$, $0 \leq u < H$, we claim

$$p^2k_u^- \equiv E_{p-1-r}k_P^-, \quad p^2k_{u+1}^+ \equiv E_rk_{P+1}^+ \pmod{p}. \quad (27)$$

For odd u write $p^2k_u^- = (pH_u^*) \sum_{z \leq u, z \text{ odd}} 2p/z$; only $z = mp$ with m odd survives, and if r is even, (26) gives $\epsilon E_r H_P^* \sum_{m \leq P} 2/m = E_{p-1-r}k_P^-$, while if r is odd the prefactor vanishes. For even u the quantities $p/(zH_z^*)$ with even $z \leq u$ are integral, so $p^2k_u^-$ is integral and vanishes mod p when r is odd; when r is even the surviving summands form complete blocks $z = mp - \lambda$, $m = 2, 4, \dots, P$, $\lambda = 0, 2, \dots, p-1$, and the downward recurrence gives $\frac{p}{(mp-\lambda)H_{mp-\lambda}^*} \equiv \frac{c_{\lambda/2}}{mH_m^*}$. With $s = (p-1)/2$, $\sum_{j=0}^s c_j^2 \equiv \binom{2s}{s} \equiv \epsilon$, and multiplying by $H_u^* \equiv E_r H_P^*$ proves the first congruence also for even u .

For the second congruence suppose first u even. In $p^2k_{u+1}^+ = H_u^* \sum_{z \equiv u(2)} 2p^2/(z^2H_z^*)$ every term with $p \nmid z$ vanishes since $v_p(H_z^*) \leq 1$; at $z = mp$, m even and $H_{mp}^* \equiv H_m^*$, giving $E_r k_{P+1}^+$ when r even and zero when r odd. For odd u write $(pH_u^*) \sum_{z \text{ odd}} 2p/(z^2H_z^*)$; only $z = mp$ survives, there $pH_{mp}^* \equiv \epsilon H_m^*$ and the two ϵ cancel. This proves the second congruence in every case and the stated local integrality.

If $i \geq j$ put $u = i - j = Pp + r$, so $d = p - 1 - r$ and $i' = j' + P$, and use $M^0(i, j) = k_u^- - \sum_{k=1}^j m_{u+k-1}/k$. After multiplication by p^2 terms with $p \nmid k$ vanish by $pm_{z-1} \equiv E_{p-1-r}m_{P-1}$, and at $k = mp$ the surviving term is $E_{p-1-r}m_{P+m-1}/m$, giving $p^2M^0(i, j) \equiv E_{p-1-r}(k_P^- - \sum_{m=1}^{j'} m_{P+m-1}/m) = E_d M^0(i', j')$. If $i < j$ put $u = j - i - 1 = Pp + r$, so $d = r$ and $i' = j' - P - 1$, and use $M^0(i, j) = k_{u+1}^+ - \sum_{k=u+2}^j m_{k-u-2}/k$. For a multiple $k = mp$, $k - u - 1 = (m - P - 1)p + (p - 1 - r)$, and the reduced summand is $E_r m_{m-P-2}/m$; a possible first multiple with $m = P + 1$ contributes zero, so the reduced sum runs over $m = P + 2, \dots, j'$ as in $M^0(i', j')$. Also $i' = \lfloor (i - \ell)/p \rfloor \geq -1$; when $i' = -1$ and $j' = P$ the sum is empty and the starting term is exactly $k_{j'+1}^+ = M^0(-1, j')$. The reduced indices are below $H/p < p/4$, so the right side is integral at p .

For Z^0 , write $I = \lfloor i/p \rfloor$ and $J = \lfloor j/p \rfloor$. Since $i, j < p^2$, $pB_i^{(1)} \equiv B_I^{(1)}$ and $p^2B_i^{(2)} \equiv B_I^{(2)}$. The diagonal formula follows at once. Off the diagonal, if $i \not\equiv j \pmod{p}$ the denominator $i - j$ is a unit and $p^2Z^0(i, j) \equiv 0$; if the residues agree, $i - j = p(I - J)$ with $I - J$ a unit, and the first harmonic congruence gives $Z^0(I, J)$. This also proves integrality of $p^2Z^0(i, j)$. $\square$

3.2 The leading layer and paired raw columns

We gather the moment reductions into column vectors. For a fixed residue $0 \leq \ell < p$ define, over $\mathbb{F}_p$,

$$P'_u = [t^{(u+1)p+\ell}] tP(t)E(t) \quad (u \geq -1), \quad D'_u = [t^{up+\ell}] D(t) \quad (u \geq 0),$$

where P, D denote the vectors of all row polynomials, with the dependence on ℓ understood. Their supports are finite. Lemma 3.1, gathered by coefficient residue, gives

$$X_{kp+\ell} := p^2F_{kp+\ell} \bmod p = \sum_{u \geq -1} P'_u M^0(u, k) - \frac{3}{2} \sum_{u \geq 0} D'_u Z^0(u, k). \quad (28)$$

For a term $[t^i]P$, its residue d satisfies $i + 1 + d = (i' + 1)p + \ell$, so it contributes exactly to $P'_{i'}$ including $i' = -1$; the Z^0 reduction survives only when $i = up + \ell$, which is the extraction defining

D'_u . The G coefficient is a sum of integral multiples of $c_{(i-j)/2}$, hence integral at every odd prime, so it disappears after multiplication by p^2 . Every raw column therefore lies in $p^{-2}\mathbb{Z}_p^n$.

If $\ell \geq H - 2p$, the degree bounds $\deg(tPE) \leq H + p - 1$ and $\deg D < H$ show P'_u, D'_u vanish for $u > 1$. Put

$$V_\ell = 2P'_1 - \frac{3}{2}D'_1, \quad U_\ell = 2P'_{-1} - \frac{3}{2}D'_0,$$

and note the small reduced values

$$\begin{array}{c|ccc} u & -1 & 0 & 1 \\ \hline M^0(u, 0) & 0 & 0 & 2 \\ M^0(u, 1) & 2 & 0 & -2 \end{array} \quad \begin{array}{c|cc} u & 0 & 1 \\ \hline Z^0(u, 0) & 0 & 1 \\ Z^0(u, 1) & 1 & -1 \end{array}.$$

Consequently, whenever the indicated columns exist,

$$X_\ell = V_\ell, \quad X_{p+\ell} = U_\ell - V_\ell. \quad (29)$$

Since tP and D have no terms below degree $A + 1$, $U_\ell = 0$ for $\ell \leq A$.

We record how changes in the full column pool are used. Let T be the n by $(n + q)$ matrix of all raw columns. If $Q \in \text{GL}_{n+q}(\mathbb{Z}_p)$ and every full minor of TQ has valuation at least $-D$, the same holds for every full minor of $T = (TQ)Q^{-1}$ by Cauchy–Binet, the coefficients being minors of the integral matrix Q^{-1} . Applying Cauchy–Binet once more to the integer filter matrix gives $v_p(\Delta_N) \geq -D$. This transfer applies even when the columns or their leading residues are linearly dependent.

For $2\sqrt{H} < p \leq H/2$ use each pair of columns with $\max(b, H - 2p) \leq \ell < \min(p, L - p, A)$. Their number is exactly

$$d_0 = (\min(p, L - p, A) - \max(b, H - 2p))_+, \quad y_+ = \max(y, 0).$$

Replacing its high column by the sum of the pair is an integral elementary column operation with integral inverse, and the pair sum lies in $p^{-1}\mathbb{Z}_p^n$ by (29), while all other columns still lie in $p^{-2}\mathbb{Z}_p^n$. The pairs are disjoint, since $\ell < p$. Any selection of n columns from the pool of $n + q$ contains at least $(d_0 - q)_+$ of the improved columns, so

$$v_p(\Delta_N) \geq -2n + (d_0 - q)_+. \quad (30)$$

3.3 The central layer and integral column elimination

For $p > H/2$ the next lemma identifies the residues of central columns after multiplication by p , in terms of the same vectors V_ℓ . The elimination retains noncentral columns and uses p times those columns to cancel the corresponding V_ℓ terms.

Lemma 3.2 (Central layer). *Suppose $H/2 < p \leq H$ and write $K = L - p$, $J = H - p$. For every central column, meaning*

$$b \leq j < \min(p, L), \quad j \geq J,$$

one has

$$pF_j \in \mathbb{Z}_p^n, \quad pF_j \equiv - \sum_{0 \leq \ell < J} \frac{V_\ell}{j - \ell} \pmod{p}. \quad (31)$$

All denominators $j - \ell$ here are units at p .

Proof. For every contributing row degree $0 \leq i < H$, $i - j \leq H - 1 - J = p - 1$ and $j - i < p$, so $|i - j| < p$ and the starting value k_{i-j}^- or k_{j-i}^+ in the diagonal reduction is integral at p . That reduction is the starting value minus $\sum_{0 \leq \ell < \min(i, j)} m_{i-\ell-1}/(j - \ell)$ with $1 \leq j - \ell < p$. By $pm_{z-1} \equiv E_{p-1-r}m_{P-1}$, a nonzero reduction after multiplication by p requires $p \leq i - \ell < 2p$, and then

$$pm_{i-\ell-1} \equiv 2E_{2p+\ell-1-i}.$$

Such an index has $0 \leq \ell < J$, and conversely its coefficient can be collected over all i without truncation since $j \geq J > \ell$ and $i \geq p + \ell > \ell$. Hence $pM^0(P, j) \equiv -\sum_{0 \leq \ell < J} 2P'_1/(j - \ell)$ with P'_1 the extracted vector of the residue ℓ . For $Z^0(i, j)$, the harmonic sums have a pole only if $i = p + \ell$ with $0 \leq \ell < J$, where the denominator $i - j$ is a unit (equality mod p would need $j = \ell < J$); then $pZ^0(p + \ell, j) \equiv 1/(\ell - j)$, and all other $Z^0(i, j)$ are integral. Combining these with the factor $-3/2$ proves (31); the G term is integral here as before. $\square$

We cancel in the full column pool, where R counts retained columns with possible denominator p^2 and S bounds the other columns with possible denominator p ; every remaining column is integral.

Lemma 3.3 (Elimination over the local integers). *For $H/2 < p \leq H$ define*

$$\begin{aligned} R &= K_+ + (K - A)_+ + (J - \max(b, K))_+, \\ S &= (\min(A, K) - b)_+ + (\min(b, J) - \max(0, K))_+. \end{aligned}$$

Then every full row minor, and hence the filtered determinant, satisfies

$$v_p(\Delta_N) \geq -\min(2n, n + R, 2R + S). \quad (32)$$

Proof. The noncentral columns are the high columns $F_{p+\ell}$ for $0 \leq \ell < K$ and the low columns F_ℓ for $b \leq \ell < J$ (empty ranges allowed: $p > b$, $J < p$, $K < J$). For $b \leq \ell < \min(A, K)$ retain the low column and replace the high column by $F_{p+\ell} + F_\ell$. Let $B = (\min(A, K) - b)_+$ be the number of these pair sums, each in $p^{-1}\mathbb{Z}_p^n$. Retain every other noncentral column as a column in $p^{-2}\mathbb{Z}_p^n$. The number retained is

$$R_0 = K_+ + (J - b)_+ - B, \quad (33)$$

which equals the stated R (split K into $K \leq 0$, $0 < K \leq b$, $b < K \leq A$, $K > A$, using $K < J$ in the last two cases).

These retained columns represent every V_ℓ with $0 \leq \ell < J$ except possibly those in

$$\mathcal{E} = \{\ell : \max(0, K) \leq \ell < \min(b, J)\}, \quad e = |\mathcal{E}| = (\min(b, J) - \max(0, K))_+.$$

For $b \leq \ell < J$ the retained low column has leading residue $p^2 F_\ell \equiv V_\ell$; for $0 \leq \ell < \min(b, K)$ the unpaired high column has leading residue $-V_\ell$ since $\ell < b < A$ makes $U_\ell = 0$. For each represented residue choose a retained column Y_ℓ and sign σ_ℓ with $p^2 Y_\ell \equiv \sigma_\ell V_\ell$, and for each central j choose $a_{\ell j} \in \mathbb{Z}_p$ reducing to $\sigma_\ell/(j - \ell)$ and replace that column by

$$C_j = F_j + p \sum_{\substack{0 \leq \ell < J \\ \ell \notin \mathcal{E}}} a_{\ell j} Y_\ell.$$

This is a shear of the whole pool with integral coefficients and integral inverse, preserving every retained column and pair sum; the resulting central columns lie in $p^{-1}\mathbb{Z}_p^n$ and $pC_j \equiv -\sum_{\ell \in \mathcal{E}} V_\ell/(j - \ell) \pmod{p}$. No information about the next coefficient of Y_ℓ is used: if $Y_\ell = p^{-2}y_0 + p^{-1}y_1 + y_2$, then $pY_\ell = p^{-1}y_0 + y_1 + py_2$ has an integral unknown second coefficient.

Let c be the number of central columns and let their scaled residues define $\varphi : \mathbb{F}_p^c \rightarrow \mathbb{F}_p^n$, $(a_j)_j \mapsto \sum_j a_j(pC_j \bmod p)$. Its image lies in the span of the e exceptional vectors, so $d := \text{rank } \varphi \leq \min(c, e)$. Choose a basis of $\mathbb{F}_p^c$ whose last $c - d$ vectors span $\ker \varphi$, lift its basis matrix to $W \in \text{Mat}_c(\mathbb{Z}_p)$; then $W \in \text{GL}_c(\mathbb{Z}_p)$ by the adjugate formula. This change leaves at most d columns in $p^{-1}\mathbb{Z}_p^n$; the other $c - d$ are integral since their scaled residues vanish and they already lay in $p^{-1}\mathbb{Z}_p^n$. If $c = 0$ this is empty.

The pool now has R columns with possible denominator p^2 , at most $B + d \leq B + e = S$ other columns with possible denominator p , and integral remaining columns. In a full minor with r columns from the first group and s from the second, the loss is at most $2r + s$, and $2r + s \leq 2n$, $2r + s \leq n + R$, $2r + s \leq 2R + S$ hold since $r + s \leq n$, $r \leq R$, $s \leq S$. All changes and inverses are integral, so Cauchy–Binet proves (32) for all original full minors and for Δ_N . $\square$

The omission of the optional pair at $\ell = A$ is harmless: its physical column is included in R . The proof also covers $K \leq 0$ and $J \leq b$; in particular, if $L \leq p \leq H$, then $R = 0$ and $S = H - p$, and at the formal endpoint $p = H$ all columns are integral.

3.4 Summing the prime losses

For a nonzero rational determinant the weighted valuations sum to its logarithmic absolute value; combining the odd-prime estimates with the bound at 2 gives the lower bound used in the final comparison.

Proposition 3.4 (Finite-place bound). *Assume $G \in \mathbb{Q}$. Along any sequence of positive integers $N \rightarrow \infty$ for which $\Delta_N \neq 0$, one has*

$$\liminf_{N \rightarrow \infty} \left(\frac{\log |\Delta_N|}{n^2} - \frac{1}{2} \log 2 \right) \geq -\frac{8609}{4608} - \left(\frac{1}{2} + \frac{505}{4608} \right) \log 2 > -2.29084. \quad (34)$$

Proof. Put $x = p/N$. For $x \leq 65/2$, (30) gives

$$\frac{d_0}{N} = (\min(x, 59 - x, 19) - \max(7, 65 - 2x))_+,$$

which is 0 for $0 \leq x \leq 23$, $2x - 46$ for $23 \leq x \leq 29$, and 12 for $29 \leq x \leq 65/2$. Only the part exceeding $q/N = 4$ improves a full minor. For $x > 65/2$ the formulas of Lemma 3.3 give the following values, retaining the intermediate breakpoint 52:

x range	R/N	S/N
$[65/2, 40]$	$105 - 2x$	12
$[40, 52]$	$65 - x$	$52 - x$
$[52, 58]$	$117 - 2x$	$x - 52$
$[58, 59]$	$59 - x$	6
$[59, 65]$	0	$65 - x$

The entries agree at common endpoints. Taking $\min(96, 48 + R/N, 2R/N + S/N)$ yields the extra breakpoint $69/2$, so for every relevant prime $p > 2\sqrt{H}$ the valuation is at least $-Nd(p/N)$ where d

is continuous, vanishes for $x \geq 65$, and has the exact pieces

x range	$d(x)$	endpoints	$\int_{\text{range}} d \, dx$
$[0, 25]$	96	96, 96	2400
$[25, 29]$	$146 - 2x$	96, 88	368
$[29, 65/2]$	88	88, 88	308
$[65/2, 69/2]$	$153 - 2x$	88, 84	172
$[69/2, 40]$	$222 - 4x$	84, 62	803/2
$[40, 58]$	$182 - 3x$	62, 8	630
$[58, 59]$	$124 - 2x$	8, 6	7
$[59, 65]$	$65 - x$	6, 0	18

(35)

In particular

$$\int_0^{65} d(x) \, dx = \frac{8609}{2}. \quad (36)$$

The omitted small odd primes cost only $o(n^2)$. At any odd prime the factorial formula gives $0 \leq v_p\binom{2l}{l} \leq 1 + \log H / \log p$ for $2l \leq H$, and the displayed formulas for $H_z^*, m_i, k_d^\pm, Z^0$ and the diagonal reduction give, with an absolute constant C_0 and with $\text{den } G$ the reduced positive denominator of G ,

$$v_p(F_j(r)) \geq -C_0 \left(1 + \frac{\log H}{\log p}\right) - v_p(\text{den } G).$$

Expanding a determinant multiplies this by at most n , and summing over $p \leq 2\sqrt{H}$ with weights $\log p$ gives $O(n\sqrt{H} \log H + n \log(\text{den } G)) = o(n^2)$. For large N every prime $p > H$ is integral throughout, so these primes contribute nothing.

We use the prime number theorem $\theta(y) = \sum_{p \leq y} \log p \sim y$ [25, Step VI, p. 707]. Fix a partition $0 = x_0 < \dots < x_m = 65$; the theorem gives $(\theta(Nx_j) - \theta(Nx_{j-1}))/N \rightarrow x_j - x_{j-1}$, and step functions from the supremum and infimum of d on each interval bound the weighted sum, whose limiting difference tends to zero as the mesh shrinks. Thus

$$\frac{1}{N} \sum_{p \leq 65N} d(p/N) \log p \longrightarrow \int_0^{65} d(x) \, dx. \quad (37)$$

The prime 2 and primes below $2\sqrt{H}$ may be removed at cost $o(1)$, using $\|d\|_\infty = 96$ and $\theta(2\sqrt{H}) \leq 2\sqrt{H} \log(2\sqrt{H})$.

Combining with (36) and $n^2 = 2304N^2$,

$$\sum_{p \text{ odd}} v_p(\Delta_N) \log p \geq -\frac{8609}{4608} n^2 - o(n^2).$$

Proposition 2.3 with $\delta = 10/48$ gives

$$v_2(\Delta_N) \geq -\left(\frac{\delta}{2} + \frac{\delta^2}{8}\right) n^2 - o(n^2) = -\frac{505}{4608} n^2 - o(n^2).$$

For a nonzero rational, $\log |\Delta_N| = \sum_p v_p(\Delta_N) \log p$; dividing by n^2 and subtracting $\frac{1}{2} \log 2$ proves (34) with $n = 48N$, so $n^2 = 2304N^2$ and $N \rightarrow \infty$ along the same sequence.

The last comparison has a short rational check. The identity $\log 2 = 2 \sum_{j \geq 0} 3^{-(2j+1)} / (2j+1)$ and its positive geometric tail give

$$\log 2 \leq 2 \sum_{j=0}^5 \frac{3^{-(2j+1)}}{2j+1} + \frac{2 \cdot 3^{-13}}{13(1-3^{-2})} < \frac{693149}{10^6}.$$

So the lower constant exceeds

$$-\frac{8609}{4608} - \frac{2809}{4608} \frac{693149}{10^6} = -\frac{10556055541}{4608000000} > -2.29084,$$

completing the proof. $\square$

4 Nonvanishing along the prime sequence

The next proposition supplies the nonzero determinants needed in Proposition 3.4.

Proposition 4.1. *Assume $G \in \mathbb{Q}$. For every sufficiently large prime p , the determinant with scale parameter $N = p$ satisfies*

$$v_p(\Delta_p) = -96p. \quad (38)$$

In particular $\Delta_p \neq 0$ for every sufficiently large prime p .

We reduce a matrix of size $48p$ to three fixed rational matrices of size 48. Throughout, an underlined row polynomial is formed with $N = 1$, so

$$n_0 = 48, \quad b_0 = 7, \quad q_0 = g_0 = 4, \quad h_0 = 2, \quad L_0 = 59, \quad C_0 = 63, \quad H_0 = 65.$$

In addition to the usual rows $0, \dots, 47$, define the auxiliary row 48 by the same formulas as in (3). For $0 \leq r \leq 48$ and $0 \leq k < 48$ put

$$\mathcal{B}(r, k) = \sum_{j=0}^4 (-1)^j \binom{4}{j} \{M^0(\underline{P}_r, 7 + k + j) - \frac{3}{2}Z^0(\underline{D}_r, 7 + k + j)\}, \quad (39)$$

with M^0, Z^0 acting linearly on the first argument. Thus $\mathcal{B}$ is a fixed rational 49×48 matrix. Let

$$\mathcal{B}_0 = (\mathcal{B}(r, k))_{0 \leq r, k < 48}, \quad \mathcal{B}_1 = (\mathcal{B}(r + 1, k))_{0 \leq r, k < 48}.$$

The certificate below proves

$$\det \mathcal{B}_0 \neq 0, \quad \det(\mathcal{B}_0 + \mathcal{B}_1) \neq 0, \quad \det(\mathcal{B}_0 - \mathcal{B}_1) \neq 0. \quad (40)$$

We first show why these three fixed facts imply the proposition.

4.1 Two palindromic bases

Let p be an odd prime and work over $\mathbb{F}_p$. The space

$$\mathcal{P}_p = \{Q \in \mathbb{F}_p[w] : \deg Q \leq 2p - 2, \ w^{2p-2}Q(1/w) = Q(w)\}$$

has dimension p . For $0 \leq \ell, i < p$ consider

$$U_\ell(w) = (2w)^\ell (1 + w^2)^{p-1-\ell}, \quad Q_i(w) = w^i \sum_{j=0}^{p-i-1} w^{2j}.$$

Both families lie in $\mathcal{P}_p$, with lowest terms $2^\ell w^\ell$ and w^i . Triangularity in degrees $0, \dots, p-1$ makes each a basis. Define $a_{\ell i} \in \mathbb{F}_p$ by $Q_i = \sum_\ell a_{\ell i} U_\ell$ and $\mathbf{a} = (a_{\ell i})$. Lowest-term comparison gives

$$a_{\ell i} = 0 \ (\ell < i), \quad a_{ii} = 2^{-i}, \quad \det \mathbf{a} = 2^{-p(p-1)/2} \neq 0, \quad (41)$$

so this varying-size matrix introduces no exceptional odd primes.

Let $Q'_0 = 0$ and $Q'_i = Q_{p-i}$ for $1 \leq i < p$, written $Q'_i = \sum_{\ell} b_{\ell i} U_{\ell}$. Then

$$\mathbf{b} = \mathbf{a}\Pi, \quad \Pi e_0 = 0, \quad \Pi e_i = e_{p-i} \quad (1 \leq i < p), \quad (42)$$

with e_i the coordinate vectors. Explicitly $Q_i = w^i(1 - w^{2(p-i)})/(1 - w^2)$ and $Q'_i = w^{p-i}(1 - w^{2i})/(1 - w^2)$. Their sum is

$$Q_i + w^p Q'_i = w^i \sum_{j=0}^{p-1} w^{2j} = w^i(1 - w^2)^{p-1},$$

since $(p-1)_j = (-1)^j$ in $\mathbb{F}_p$. With $t = 2w/(1+w^2)$, $f = (1-w^2)/(1+w^2)$, $E(t) = (1-t^2)^{(p-1)/2} = f^{p-1}$, division by $(1+w^2)^{p-1}$ gives the identity

$$E(t)w^i = \sum_{\ell=0}^{p-1} t^{\ell} (a_{\ell i} + b_{\ell i} w^p). \quad (43)$$

4.2 Frobenius and the two extraction shifts

Set $N = p$, write each row index uniquely as $r = pr_0 + i$ with $0 \leq r_0 < 48$ and $0 \leq i < p$, and put $t' = t^p$, $w' = w^p$, $f' = f^p$. Frobenius gives $t' = 2w'/(1+(w')^2)$, $f' = (1-(w')^2)/(1+(w')^2)$ and $(f')^2 = 1 - (t')^2$. With $C = 63p$, $h = 2p$, $g = 4p$ in (3),

$$\begin{aligned} tR_r E(t) &= (1-t')^2 (t')^{63} (w')^{r_0-4} E(t) w^i \\ &= \sum_{\ell=0}^{p-1} t^{\ell} t' (a_{\ell i} \underline{R}_{r_0}(t', w') + b_{\ell i} \underline{R}_{r_0+1}(t', w')). \end{aligned} \quad (44)$$

Here the single factor w' in the second term creates one successor row of index at most 48. Since star fixes t, t' and negates f, f' , and $fE = f'$, the row identities give $tR_r E = tP_r E - f'D_r$ and $t'\underline{R}_j = t'\underline{P}_j - f'\underline{D}_j$. Taking the star-invariant part of (44), then its anti-invariant part and cancelling f' , yields

$$tP_r(t)E(t) = \sum_{\ell=0}^{p-1} t^{\ell} t' (a_{\ell i} \underline{P}_{r_0}(t') + b_{\ell i} \underline{P}_{r_0+1}(t')), \quad (45)$$

$$D_r(t) = \sum_{\ell=0}^{p-1} t^{\ell} (a_{\ell i} \underline{D}_{r_0}(t') + b_{\ell i} \underline{D}_{r_0+1}(t')). \quad (46)$$

These are polynomial identities, since both sides are polynomials and $t = 2w/(1+w^2)$ is injective on $\mathbb{F}_p[t]$.

In the notation of (28) the extractions for a fixed residue ℓ are $P'_u = [t^{(u+1)p+\ell}] tP_r E$ and $D'_u = [t^{up+\ell}] D_r$. Every polynomial is uniquely $\sum_{\ell} t^{\ell} A_{\ell}(t^p)$, so the last two identities give

$$(P'_u, D'_u) = [(t')^u] \{a_{\ell i}(\underline{P}_{r_0}, \underline{D}_{r_0})(t') + b_{\ell i}(\underline{P}_{r_0+1}, \underline{D}_{r_0+1})(t')\}. \quad (47)$$

The t' in the first identity is exactly consumed by the $u+1$ shift in the P extraction; the D extraction has none, and $P'_{-1} = 0$ here.

There is a degree issue at the last successor row. The auxiliary polynomials have

$$\min \deg \underline{P}_{48} = 62 - 44 = 18, \quad \min \deg \underline{D}_{48} = 63 - 44 = 19. \quad (48)$$

For the last block $r = 47p + i$, $i > 0$, (42) gives $b_{\ell i} = 0$ for $\ell < p - i$ and $b_{p-i, i} = 2^{-(p-i)} \neq 0$, so the successor contribution starts at $19p + (p - i) = 20p - i$, while the first-row contribution starts no earlier than $20p + i$. Directly from the rows, $|r - g| = 43p + i$ makes tP_r and D_r start at $C - |r - g| = 20p - i$, matching the extraction since $E(0) = 1$. The leading Chebyshev coefficients are powers of 2, nonzero in odd characteristic; when $i = 0$ the successor term vanishes and the first term starts at $20p$. Thus the lower degree of the auxiliary row injects no forbidden coefficient, and its contact $tR_{48}/f = O(t^{107})$ holds since $63 + 48 - 4 = 107 > 59$.

4.3 The residue blocks and their determinant

Take $p > 260$ and exclude the primes dividing the denominator of the hypothetical rational G . Then $p > 2\sqrt{65p} = 2\sqrt{H}$, so (24) and (28) apply at this same prime. The rational row entry is

$$F_j(r) = M^0(P_r, j) - \frac{3}{2}Z^0(D_r, j) + 4G \sum_v [t^v]P_r c_{(v-j)/2},$$

whose last sum is p -integral (integer polynomial coefficients, and every c_d integral at odd p), so the G term vanishes after multiplication by p^2 . All scaled entries are p -integral by the cited layer formulas. Substituting (47) into (28) gives

$$\begin{aligned} p^2 F_{kp+\ell}(pr_0 + i) &\equiv a_{\ell i} \{M^0(\underline{P}_{r_0}, k) - \frac{3}{2}Z^0(\underline{D}_{r_0}, k)\} \\ &\quad + b_{\ell i} \{M^0(\underline{P}_{r_0+1}, k) - \frac{3}{2}Z^0(\underline{D}_{r_0+1}, k)\} \pmod{p}. \end{aligned} \quad (49)$$

The base entries have denominators with prime factors at most 65, so their reductions exist for $p > 260$.

The filter respects residues: writing $k = \ell + pk_0$ with $0 \leq \ell < p$ and $0 \leq k_0 < 48$,

$$s^{7p+k}(1-s)^{4p} = s^\ell (s^p)^{7+k_0} (1-s^p)^4 \quad \text{in } \mathbb{F}_p[s]. \quad (50)$$

Multiplying raw entries by p^2 before reduction makes this a valid column operation (coefficient differences divisible by p multiply integral scaled entries). The fourth difference uses exactly the five base indices $7 + k_0, \dots, 11 + k_0$, all in $7, \dots, 58$. Each residue group thus holds exactly 48 base columns, and no operation changes ℓ .

Order rows by (i, r_0) and columns by (ℓ, k_0) . The reduction of the scaled $48p \times 48p$ matrix has (i, ℓ) block $a_{\ell i}\mathcal{B}_0 + b_{\ell i}\mathcal{B}_1$. Writing this matrix as $\mathcal{L}_p$,

$$\begin{aligned} \mathcal{L}_p &= \mathbf{a}^T \otimes \mathcal{B}_0 + \mathbf{b}^T \otimes \mathcal{B}_1, \\ \mathcal{L}_p((\mathbf{a}^T)^{-1} \otimes I_{48}) &= I_p \otimes \mathcal{B}_0 + \Pi^T \otimes \mathcal{B}_1, \end{aligned} \quad (51)$$

using $\mathbf{b}^T = \Pi^T \mathbf{a}^T$.

The nonzero indices $1, \dots, p-1$ split into $(p-1)/2$ pairs $\{i, p-i\}$, on which $e_i + e_{p-i}$ and $e_i - e_{p-i}$ are eigenvectors of Π with eigenvalues $1, -1$; with e_0 of eigenvalue 0 they form a basis since 2 is invertible. Thus Π^T has eigenvalue multiplicities $1, (p-1)/2, (p-1)/2$ for $0, 1, -1$. A similarity on the p -dimensional factor proves the exact block identity

$$\begin{aligned} \det \mathcal{L}_p &= (\det \mathbf{a})^{48} \det \mathcal{B}_0 \\ &\quad \cdot \det(\mathcal{B}_0 + \mathcal{B}_1)^{(p-1)/2} \det(\mathcal{B}_0 - \mathcal{B}_1)^{(p-1)/2}. \end{aligned} \quad (52)$$

There is no factor from the similarity; returning to the original orders changes at most the sign.

4.4 An exact certificate for the fixed matrices

Every entry of (39) is constructed by the following rational arithmetic, which also fixes its reduction mod 101. Form $m_0 = 2$, $m_i = 0$ for odd i , and $m_i = im_{i-2}/(i+1)$ for even $2 \leq i \leq 64$. Set $k_0^- = 0$, $k_1^- = 2$, $k_0^+ = k_1^+ = 0$ and

$$\begin{aligned} k_d^- &= \frac{(d-1)k_{d-2}^- + m_{d-2} + m_{d-1}}{d} & (2 \leq d \leq 64), \\ k_d^+ &= \frac{(d-2)k_{d-2}^+ + 2/(d-1)}{d-1} & (2 \leq d \leq 58). \end{aligned}$$

For $0 \leq i \leq 64$, $0 \leq j \leq 58$ form $Y_{i0} = k_i^-$, $Y_{0j} = k_j^+$, $Y_{ij} = Y_{i-1,j-1} - m_{i-1}/j$ ($i, j \geq 1$), so $Y_{ij} = M^0(i, j)$. With $B_i^{(d)} = \sum_{u=1}^i u^{-d}$ and $B_0^{(d)} = 0$, set

$$J_{ij} = \frac{3}{2} \begin{cases} -B_i^{(2)}, & i = j, \\ (B_i^{(1)} - B_j^{(1)})/(i-j), & i \neq j, \end{cases}$$

so $J_{ij} = \frac{3}{2}Z^0(i, j)$ with its diagonal sign included.

For the polynomials define $E_0 = t^{62}$, $E_1 = t^{61}$, $I_0 = 0$, $I_1 = t^{62}$ and, for either family $S = E, I$, $S_m = 2S_{m-1}/t - S_{m-2}$ ($2 \leq m \leq 44$). These integer polynomials are $E_m = t^{62}T_m(1/t)$ and $I_m = t^{62}U_m(1/t)$. For $0 \leq r \leq 48$ let $u = |r-4|$ and form, in degrees $0, \dots, 64$,

$$y_r(t) = (1-t)^2 E_u(t), \quad z_r(t) = \text{sign}(r-4)(1-t)^2 I_u(t).$$

They are exactly $\underline{P}_r$ and $\underline{D}_r$. Contract to the length-52 vector

$$v_r(j) = \sum_{i=0}^{64} ([t^i]y_r Y_{ij} - [t^i]z_r J_{ij}), \quad j = 7, \dots, 58.$$

Replacing v four times successively by $(v(j) - v(j+1))_j$ gives the length-48 row r of $\mathcal{B}$, including the auxiliary row's degree-18 term.

Every scalar denominator here is a product of nonzero integers of absolute value at most 65, so each is a unit mod 101; polynomial division by t only shifts exponents of polynomials divisible by t . Hence the whole construction runs over $\mathbb{F}_{101}$ and agrees with reduction of the rational matrices.

The elimination rule is: for $\sigma \in \{0, 1, -1\}$ start with $R_i = \mathcal{B}(i, \cdot) + \sigma \mathcal{B}(i+1, \cdot)$, $0 \leq i < 48$, in increasing order; at step i , if $R_i(i) = 0$, swap row i with the first later row having a nonzero entry in column i ; record $d_i = R_i(i)$ and replace every row $j > i$ by $R_j - (R_j(i)/d_i)R_i$. There are no swaps for $\sigma = 0, -1$; for $\sigma = 1$ the only swaps (0-based) are (30, 31) and (45, 46). The pivots over $\mathbb{F}_{101}$ are, 16 per line:

σ	Successive pivots in $\mathbb{F}_{101}$
0	60, 68, 62, 79, 47, 32, 69, 57, 30, 72, 35, 66, 43, 20, 85, 48 88, 4, 77, 54, 60, 79, 26, 68, 83, 39, 40, 65, 1, 68, 78, 24 15, 98, 32, 22, 94, 9, 99, 10, 15, 75, 4, 2, 25, 53, 90, 79
1	38, 51, 90, 70, 5, 21, 88, 55, 45, 20, 35, 41, 77, 10, 18, 25 76, 14, 38, 72, 6, 66, 56, 35, 83, 58, 56, 11, 17, 20, 30, 24 28, 11, 25, 70, 79, 99, 66, 38, 4, 41, 63, 91, 63, 17, 90, 98
-1	82, 92, 26, 87, 21, 74, 87, 88, 88, 3, 14, 23, 38, 58, 36, 20 26, 33, 94, 74, 78, 45, 93, 86, 73, 66, 45, 30, 61, 3, 88, 27 20, 58, 69, 48, 78, 39, 48, 1, 66, 18, 86, 93, 52, 92, 39, 77

Every listed pivot is nonzero, so the swaps and row additions are invertible and all three matrices are invertible mod 101, hence have nonzero determinants over $\mathbb{Q}$; this proves (40). The modulus 101 serves only this fixed certificate; the growing determinant uses arbitrary large primes as follows.

Completion of the proof of Proposition 4.1. Let $\mathcal{E}$ consist of 2, the primes dividing the denominator of G , the primes dividing any denominator of an entry of $\mathcal{B}$, and the primes dividing the numerator or denominator of the three nonzero rational determinants in (40); this is a fixed finite set. For $p > 260$ outside $\mathcal{E}$ the three fixed matrices reduce to invertible matrices over $\mathbb{F}_p$, and (41) together with (52) give $\det \mathcal{L}_p \neq 0$.

Let $\mathcal{F}_p$ be the original filtered matrix with determinant Δ_p , of size $n = 48p$; then $p^2 \mathcal{F}_p$ is integral over $\mathbb{Z}_{(p)}$, and its reduction after the stated row and column permutations is $\mathcal{L}_p$. Hence $p^{2n} \Delta_p = \det(p^2 \mathcal{F}_p)$ is a unit in $\mathbb{Z}_{(p)}$, so $v_p(\Delta_p) = -2n = -96p$. Every sufficiently large prime lies outside $\mathcal{E}$ and exceeds 260. $\square$

5 The real-place determinant estimates

We estimate the determinant (7) without any rationality assumption. Write

$$\alpha = \frac{a}{n} = \frac{11}{48}, \quad \beta = \frac{b}{n} = \frac{7}{48}, \quad \gamma = \frac{g}{n} = \frac{q}{n} = \frac{4}{48}, \quad \eta = \frac{h}{n} = \frac{2}{48}, \quad D_* = n - 1 - 2g = 40N - 1, \quad (53)$$

so $0 < D_* < n$ and $n \geq 48$. For a real list y of length n let $V(y) = \prod_{i < j} (y_j - y_i)$ and $\mathcal{V}(y) = |V(y)|$. All constants below are independent of the locations and separations of the integration variables.

5.1 The integral and its two sheets

Put $d\mu(t) = |t| dt / f(t)$ on $(-1, 1)$ and set $A_r(t) = P_r(t) - \frac{3}{2} \mathbf{1}_{\{t > 0\}} \frac{f(t)}{t} D_r(t)$ and $\psi_k(s) = s^{b+k} (1-s)^q$. The value at $t = 0$ is immaterial; the combination has a finite limit there because $D_r(t)/t$ is a polynomial. The determinant entries are exactly $\int_{-1}^1 \int_0^1 A_r(t) \psi_k(s) / (1-ts) ds d\mu(t)$. The measure has mass $\mu((-1, 1)) = 2$, and the scalar majorant $\iint (1 - |t|s)^{-1} ds d\mu(t)$ is finite since integration in s gives at worst an integrable logarithmic singularity at $|t| = 1$; the functions A_r, ψ_k are bounded for fixed n . Thus all permutation expansions below are absolutely integrable.

Applying Andréief's identity twice [10, Equation (1.7) and Section 2.2], expanding determinants and relabelling,

$$\Delta_N = \frac{1}{(n!)^2} \int_{(-1,1)^n} \int_{(0,1)^n} \det[A_r(t_i)] \det\left[\frac{1}{1-t_i s_j}\right] \det[\psi_k(s_j)] ds d\mu^n(t). \quad (54)$$

Fubini applies by the majorant. Cauchy's double alternant [12, Section 2.1, Equation (2.7)] is

$$\det\left[\frac{1}{1-t_i s_j}\right] = \frac{V(t)V(s)}{\prod_{i,j} (1-t_i s_j)}; \quad (55)$$

multiplying by the denominator gives a polynomial alternating separately in t and s , and expanding at $s = 0$ shows the constant is 1. Also $\det[\psi_k(s_j)] = V(s) \prod_j s_j^b (1-s_j)^q$.

Use the real coordinate

$$x_i = \frac{t_i}{1 + \sqrt{1-t_i^2}}, \quad t_i = \frac{2x_i}{1+x_i^2}, \quad d\mu(t_i) = \frac{4|x_i|}{(1+x_i^2)^2} dx_i, \quad (56)$$

which maps $(-1, 1)$ bijectively to $(-1, 1)$. Except on a null set we may assume the x_i are distinct and nonzero. Define

$$(\xi_f(x), \xi_n(x)) = \begin{cases} (1/2, 1/2), & x < 0, \\ (-1/4, 5/4), & x > 0, \end{cases} \quad \mathcal{R}_n(x) = \det[\xi_f(x_i)x_i^{g-r} + \xi_n(x_i)x_i^{r-g}].$$

The subscripts refer to evaluation at x_i (near) and $1/x_i$ (far), inside and outside the unit circle. Since $fD_r/t = (R_r^* - R_r)/2$, the negative half-interval has row $(R_r + R_r^*)/2$ and the positive half has $(5R_r - R_r^*)/4$; factoring $(1 - t_i)^h t_i^{C-1}$ from column i gives the exact identity

$$\Delta_N = \frac{1}{(n!)^2} \int \int \mathcal{R}_n(x) \frac{V(t)V(s)^2}{\prod_{i,j}(1 - t_i s_j)} \prod_j s_j^b (1 - s_j)^q \prod_i t_i^{C-1} (1 - t_i)^h ds d\mu^n(t). \quad (57)$$

There is one t -Vandermonde and two s -Vandermondes, and all denominators are positive.

5.2 A uniform interpolating function

Put $c_i = \xi_n(x_i)/\xi_f(x_i)$, nonzero on both halves. For $0 \leq r < n$,

$$\xi_f(x_i)x_i^{g-r} + \xi_n(x_i)x_i^{r-g} = \xi_f(x_i)x_i^{g-(n-1)}(x_i^{n-1-r} + c_i x_i^{D_*} x_i^r). \quad (58)$$

Thus a holomorphic h_* with $h_*(x_i) = c_i x_i^{D_*}$ turns the bracket into $x_i^{n-1-r} + h_*(x_i)x_i^r$, and we bound the resulting evaluation determinant through the norm of h_* . The first estimate applies when

$$\sum_{i=1}^n \frac{1 - x_i^2}{1 + x_i^2} \leq D_*. \quad (59)$$

The construction is uniform even when the nodes cluster.

Lemma 5.1. *Let $x_1, \dots, x_n$ be distinct nonzero reals in $(-1, 1)$ satisfying (59). There is a function h_* holomorphic on a neighborhood of the closed unit disk with*

$$h_*(x_i) = c_i x_i^{D_*}, \quad c_i = \begin{cases} 1, & x_i < 0, \\ -5, & x_i > 0, \end{cases} \quad \sup_{|z| \leq 1} |h_*(z)| \leq K_0 n, \quad K_0 = 10e^{12}.$$

The neighborhood may depend on the nodes; the bound does not.

Proof. Set $B(z) = \prod_i (z - x_i)/(1 - x_i z)$ and $F(z) = z^{D_*}/B(z)$. For $0 < u \leq 1$ and real $0 < |x| < 1$,

$$\frac{d}{d \log u} \frac{1}{2} \log \frac{u^2 + x^2}{1 + x^2 u^2} = \frac{(1 - x^4)u^2}{(u^2 + x^2)(1 + x^2 u^2)} \leq \frac{1 - x^2}{1 + x^2},$$

using $u^2 + u^{-2} \geq 2$. Integrating from u to 1 gives $|B(iu)| \geq u^{\sum_i (1 - x_i^2)/(1 + x_i^2)} \geq u^{D_*}$; the same holds at $-iu$, and $F(0) = 0$. For $1 \leq u \leq 1 + 2/n$ each factor has modulus at least one because $(u^2 + x^2) - (1 + x^2 u^2) = (u^2 - 1)(1 - x^2) \geq 0$, so

$$|F(iu)| \leq e^2 \quad (|u| \leq 1 + 2/n). \quad (60)$$

Orient $\Gamma = [-i(1 + 2/n), i(1 + 2/n)]$ upwards and define, off Γ , $\mathcal{C}(z) = \frac{1}{2\pi i} \int_{\Gamma} 6F(w)/(w - z) dw$. Write $c_- = 1$, $c_+ = -5$ and

$$h_*(z) = c_{\pm} z^{D_*} - B(z)\mathcal{C}(z) \quad \text{for } \pm \operatorname{Re} z > 0. \quad (61)$$

The density is holomorphic near Γ (its poles x_i are nonzero real). Local deformation shows the left minus right lateral value of $\mathcal{C}$ equals $6F$: moving a short upward piece of Γ to its left gives a positively oriented contour with residue $6F(z)$. Hence the two continuations agree, because $(c_- - c_+)z^{D_*} - 6B(z)F(z) = 0$; this glues h_* holomorphically across Γ , including its two crossings of the unit circle. For each fixed node list the poles $1/x_i$ and the endpoints of Γ lie strictly outside the closed disk, so a small exterior neighborhood avoids them and the same gluing applies; h_* is holomorphic there, and since $B(x_i) = 0$ and $\mathcal{C}$ is regular at the nonzero real x_i , the interpolation follows.

For the bound, in either radius-1/4 disk about $\pm i$ the quantities $|w|, |w - x_i|, |1 - x_i w|$ are at least 3/4, so

$$|F'(w)/F(w)| \leq \frac{4}{3}D_* + \frac{8}{3}n < 4n. \quad (62)$$

As $|F(\pm i)| = 1$, integration within those disks gives $|F(w)| \leq e^{12}$ whenever $|w \mp i| \leq 3/n$. If $|z| = 1$ and $|\operatorname{Re} z| \geq 1/(10n)$, the segment is at distance at least $1/(10n)$ and length at most 3, so (60) gives $|\mathcal{C}(z)| \leq 30e^2n$. If $0 < |\operatorname{Re} z| < 1/(10n)$, z is near $\pm i$; replacing the relevant portion of Γ by three sides of a rectangle whose far vertical side has real part $-\operatorname{sign}(\operatorname{Re} z)/n$ (in the radius-1/4 disk, containing neither z nor a pole) leaves $\mathcal{C}(z)$ unchanged. Each new side is within $3/n$ of $\pm i$; the contour has length at most 3 and distance at least $1/(2n)$ from z (the vertical side by $\geq 1/n$, the outer horizontal side by $\geq 2/n$, the inner horizontal side and remaining diameter by $1/n - (1 - \sqrt{1 - (10n)^{-2}}) > 1/(2n)$). Thus $|\mathcal{C}(z)| \leq 6e^{12}n$ here. Finally $|B(z)| = 1$ on $|z| = 1$, so (61) gives $|h_*(z)| \leq K_0n$ there, and at $z = \pm i$ by continuity; the maximum modulus principle extends the bound. $\square$

5.3 Evaluation in a finite-dimensional Hilbert space

The next balance uses the kernel and compression viewpoint of bounded analytic interpolation [21]; the finite-dimensional facts are proved directly.

Proposition 5.2 (Interpolation estimate). *Under (59),*

$$|\mathcal{R}_n(x)| \leq (1 + K_0n)^n \mathcal{V}(1/x) \prod_i |x_i|^g. \quad (63)$$

The prefactor is $\exp(O(n \log n))$, uniformly in the nodes.

Proof. Let $Q(z) = \prod_i (1 - x_i z)$ and give $\mathcal{H}_Q = \{v/Q : v \in \mathbb{C}[z], \deg v < n\}$ the H^2 norm $\|v/Q\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |v(e^{i\theta})|^2 / |Q(e^{i\theta})|^2 d\theta$. The zeros of Q lie outside the closed disk, so this is a genuine norm. Define $\tilde{v}(z) = z^{n-1}v(1/z)$ and $R(v/Q) = \tilde{v}/Q$; this is a complex-linear isometry, since $|\tilde{v}(e^{i\theta})| = |v(e^{-i\theta})|$ and $|Q(e^{i\theta})| = |Q(e^{-i\theta})|$.

The functions $k_i(z) = (1 - x_i z)^{-1}$ lie in $\mathcal{H}_Q$, are linearly independent (distinct poles $1/x_i$), and form a basis; for $u \in H^2$ the Taylor inner product gives $\langle u, k_i \rangle = u(x_i)$, so the orthogonal projection Π_Q onto $\mathcal{H}_Q$ satisfies $(\Pi_Q u)(x_i) = u(x_i)$. Multiplication by h_* has H^2 operator norm at most $\|h_*\|_\infty$, so $J = \Pi_Q M_{h_*}|_{\mathcal{H}_Q}$ and $L = I + JR$ satisfy $\|L\| \leq 1 + K_0n$.

Let $E : \mathcal{H}_Q \rightarrow \mathbb{C}^n$ send v/Q to $(v(x_i))_i$; it is invertible since a degree- $< n$ polynomial cannot vanish at all distinct nodes, and $E(u)_i = Q(x_i)u(x_i)$. Because $LR = R + J$ and projection preserves values, applying this to $LR(v/Q)$ cancels the factors $Q(x_i)$ and gives

$$ELR(v/Q) = (\tilde{v}(x_i) + h_*(x_i)v(x_i))_i. \quad (64)$$

In the basis $1/Q, \dots, z^{n-1}/Q$, $\det E = V(x)$ and $|\det R| = 1$, so the determinant of the mixed evaluations divided by $V(x)$ has absolute value $|\det L| \leq \|L\|^n \leq (1 + K_0n)^n$. This is an algebraic

cancellation of the evaluation determinant; no bound for the inverse evaluation matrix enters. As $|\xi_f(x_i)| \leq 1/2 \leq 1$ and $\mathcal{V}(1/x) = \mathcal{V}(x) \prod_i |x_i|^{-(n-1)}$, (63) follows from (58). $\square$

Proposition 5.3 (Hadamard estimate). *For every distinct nonzero real list in $(-1, 1)$, in particular when (59) fails,*

$$|\mathcal{R}_n(x)| \leq (3/2)^n n^{n/2} 2^{-\binom{n}{2}} \prod_i (1 + x_i^2)^{(n-1)/2} |x_i|^{g-(n-1)}. \quad (65)$$

Proof. Expand the determinant by choosing one sheet in each column. For $z_i \in \{x_i, 1/x_i\}$ the absolute monomial determinant is $\mathcal{V}(z) \prod_i |z_i|^{-g}$. Write $z_i = \tan \phi_i$, $-\pi/2 < \phi_i < \pi/2$; then

$$\mathcal{V}(z) = \prod_i (1 + z_i^2)^{(n-1)/2} \prod_{i < j} |\sin(\phi_j - \phi_i)|.$$

The last product is $2^{-\binom{n}{2}}$ times the Vandermonde on the unit-circle points $e^{2i\phi_i}$, whose monomial matrix has column norms $\sqrt{n}$, so Hadamard gives $2^{-\binom{n}{2}} n^{n/2}$. The far-to-near weight ratio is $|1/x|^{-g}(1+x^{-2})^{(n-1)/2}/(|x|^{-g}(1+x^2)^{(n-1)/2}) = |x|^{-D_*} \geq 1$, so all weights are bounded by their far-sheet values; and the sum of absolute sheet coefficients is at most $\prod_i (|\xi_f(x_i)| + |\xi_n(x_i)|) \leq (3/2)^n$. $\square$

5.4 The two principal integrands

Put $t_i = 2x_i/(1+x_i^2)$ and

$$\mathcal{J}_n(x, s) = \frac{\mathcal{V}(t)\mathcal{V}(s)^2}{\prod_{i,j}(1-t_i s_j)} \prod_j s_j^b (1-s_j)^q \prod_i |t_i|^{C-1} (1-t_i)^h.$$

Define

$$\begin{aligned} \mathcal{I}_{2,n}(x, s) &= \mathcal{J}_n(x, s) \mathcal{V}(1/x) \prod_i |x_i|^g, \\ \mathcal{I}_{1,n}(x, s) &= \mathcal{J}_n(x, s) 2^{-\binom{n}{2}} \prod_i (1+x_i^2)^{(n-1)/2} |x_i|^{g-(n-1)}. \end{aligned} \quad (66)$$

Let $\Omega_{2,n}$ be the interior configurations with distinct nonzero x_i satisfying (59), and $\Omega_{1,n}$ the complementary case. We use the interpolation estimate on $\Omega_{2,n}$ and the Hadamard estimate on $\Omega_{1,n}$ (valid in both cases). After $t = 2x/(1+x^2)$, the index κ is the exponent of the absolute x -Vandermonde in the majorant: $\kappa = 2$ for interpolation, $\kappa = 1$ for Hadamard; the s_j range over $(0, 1)$.

The propositions and (57) give

$$|\Delta_N| \leq \frac{2^n K_n}{(n!)^2} \max_{\kappa \in \{1, 2\}} \sup_{(x, s) \in \Omega_{\kappa, n}} \mathcal{I}_{\kappa, n}(x, s), \quad K_n = \max\{(1 + K_0 n)^n, (3/2)^n n^{n/2}\},$$

since $d\mu^n(t)ds$ has total mass 2^n and the two case sets partition the domain up to a null set. In logarithmic form, with $\log 0 = -\infty$,

$$\frac{\log |\Delta_N|}{n^2} - \frac{1}{2} \log 2 \leq \max_{\kappa \in \{1, 2\}} \sup_{(x, s) \in \Omega_{\kappa, n}} \left(\frac{\log \mathcal{I}_{\kappa, n}(x, s)}{n^2} - \frac{1}{2} \log 2 \right) + O\left(\frac{\log(n+2)}{n}\right). \quad (67)$$

The error is independent of the integration lists; factorials, measure masses and sheet sums are all accounted for.

Repeated nonzero x_i give a zero row determinant and are handled by continuity in the first estimate; repeated s_j give a zero s Vandermonde. Sets with a zero or boundary node have measure zero, and near them the original expression (54) supplies the integrable definition. No separation condition is needed in any bound, so all such configurations remain in the suprema of (67).

6 A uniform energy bound

This section replaces the two many-variable majorants by one-variable suprema plus explicit quadratic terms. The bound must be uniform before taking those suprema; matched damping and retained endpoint corrections provide this uniformity.

We retain $\alpha, \beta, \gamma, \eta$, the cases $\kappa = 2, 1$, and the nonnegative majorants $\mathcal{I}_{\kappa, n}$ of (66). Put

$$\begin{aligned} W_\kappa(x) &= (\alpha + 2\gamma) \log |x| + 2\eta \log(1 - x) - (\kappa/2 + \alpha + \eta + \gamma) \log(1 + x^2), \\ W_s(s) &= \beta \log s + \gamma \log(1 - s), \\ D(x) &= 2\gamma - \frac{2x^2}{1 + x^2}. \end{aligned} \quad (68)$$

Here $D(x)$ is a scalar function, distinct from the row polynomials D_r . For a real sequence $u = (u_k)_{k \geq 1}$ with $|u_k| \leq Kr^k$ for some $K < \infty$ and $0 < r < 1$, define

$$\|u\|_*^2 = \sum_{k \geq 1} \frac{u_k^2}{k}, \quad T(u, x) = \sum_{k \geq 1} \frac{u_k T_k(x)}{k}, \quad S(u, x) = \sum_{k \geq 1} \frac{u_k x^k}{k}, \quad (69)$$

uniformly convergent for $-1 \leq x \leq 1$. The sequences p, v below are freely chosen trial coefficients; the inequality $-a^2 \leq b^2 - 2ab$ replaces two negative quadratic sums by affine upper bounds, leaving two one-variable suprema. Each admissible choice gives a valid bound, and the certificate supplies one successful choice per case.

Proposition 6.1. *Let p, v be two such sequences and take $\lambda = 0$ in case $\kappa = 2$ or any fixed $\lambda \geq 0$ in case $\kappa = 1$. Uniformly over configurations in the indicated case,*

$$\begin{aligned} \limsup_{N \rightarrow \infty} \sup_{\text{case } \kappa} \left(\frac{\log \mathcal{I}_{\kappa, n}}{n^2} - \frac{1}{2} \log 2 \right) \\ \leq (-1 + \alpha + \gamma) \log 2 + \kappa \|p\|_*^2 + \frac{1}{2} \|v\|_*^2 \\ + \sup_{\substack{-1 \leq x < 1 \\ x \neq 0}} \{W_\kappa(x) + \lambda D(x) - 2\kappa T(p, x) - S(v, x)\} \\ + \sup_{0 < s < 1} \{W_s(s) + 2T(v, s)\}. \end{aligned} \quad (70)$$

Consequently the maximum of the right sides for the two cases bounds $\limsup_{N \rightarrow \infty} (n^{-2} \log |\Delta_N| - \frac{1}{2} \log 2)$. We use $\log 0 = -\infty$.

Proof. Write $\langle F(x) \rangle = n^{-1} \sum_i F(x_i)$, and similarly for s ; in a double average $\langle K(x, x') \rangle_{\neq}$ sum over ordered pairs $i \neq j$ and divide by n^2 . The substitution $t = 2x/(1 + x^2)$ gives

$$|t - t'| = \frac{2|x - x'| (1 - xx')}{(1 + x^2)(1 + x'^2)}, \quad 1 - ts = \frac{1 - 2xs + x^2}{1 + x^2}, \quad 1 - t = \frac{(1 - x)^2}{1 + x^2}. \quad (71)$$

The exponent of $|x_i|$ in either majorant is exactly $(C - 1) + g - (n - 1) = C + g - n = a + 2g$, so there is no singular finite-size correction at $x = 0$. Keeping the other powers gives the exact identity

$$\begin{aligned} \frac{\log \mathcal{I}_{\kappa, n}}{n^2} - \frac{1}{2} \log 2 &= c_{\kappa, n} \log 2 + \left\langle W_\kappa(x) + \frac{\kappa + 2}{2n} \log(1 + x^2) \right\rangle + \langle W_s(s) \rangle \\ &\quad + \frac{\kappa}{2} \langle \log |x - x'| \rangle_{\neq} + \frac{1}{2} \langle \log(1 - xx') \rangle_{\neq} + \langle \log |s - s'| \rangle_{\neq} \\ &\quad - \langle \log(1 - 2xs + x^2) \rangle, \\ c_{\kappa, n} &= \frac{C}{n} + \frac{\kappa - 2}{2} - \frac{\kappa + 1}{2n}. \end{aligned} \quad (72)$$

The correction $(\kappa + 2)/(2n)\langle \log(1 + x^2) \rangle$ is bounded by $2 \log 2/n$.

We use the classical Chebyshev expansion of the logarithmic kernel, in the form of Haagerup's identity [11, Lemma 3.1 and its proof]. On $[-1, 1]$, with $u = \cos \theta$, $u' = \cos \theta'$, $0 \leq r < 1$,

$$\begin{aligned} L_r(u, u') &= -\log 2 + \log |1 - re^{i(\theta+\theta')}| + \log |1 - re^{i(\theta-\theta')}| \\ &= -\log 2 - 2 \sum_{k \geq 1} \frac{r^k T_k(u) T_k(u')}{k}. \end{aligned} \quad (73)$$

Letting $r \uparrow 1$ at distinct u, u' gives $L_1(u, u') = \log |u - u'|$, since the product of the two chord lengths is $2|u - u'|$. For $|z|, |z'| < 1$,

$$\log(1 - zz') = - \sum_{k \geq 1} \frac{z^k z'^k}{k}, \quad -\log(1 - 2zs + z^2) = 2 \sum_{k \geq 1} \frac{z^k T_k(s)}{k}, \quad (74)$$

the last being the log of $|1 - ze^{i \arccos s}|^2$. The singular kernels cannot be summed at diagonals, so we regularize all appearances of the same moment by the same factor, preserving completion of squares. Near $(x, s) = (1, 1)$, $|1 - xe^{i \arccos s}|^2 = (1 - x)^2 + 2x(1 - s)$ identifies the scales $1 - x$ and $\sqrt{1 - s}$.

Fix $0 < \varepsilon < 1/8$ and define

$$\tau = 1 - \varepsilon, \quad \rho(x) = 1 - \varepsilon(1 - x), \quad \sigma(s) = 1 - \varepsilon\sqrt{1 - s}. \quad (75)$$

Replace the x cosine kernel by $L_{\tau^2}(x, x')$, the s cosine kernel by $L_{\sigma(s)\sigma(s')}(s, s')$, the power kernel by $\log(1 - \rho(x)\rho(x')xx')$, and the cross kernel by $-\log |1 - \rho(x)\sigma(s)xe^{i \arccos s}|^2$. Each original kernel is at most its replacement plus $O(\varepsilon)$, with an absolute constant.

For $|\zeta| = 1$ and $0 < r \leq 1$, $|1 - r\zeta|^2 - r|1 - \zeta|^2 = (1 - r)^2 \geq 0$, so each cosine kernel costs at most $-\log r \leq O(\varepsilon)$. For the power kernel put $u = xx'$ and $r = \rho(x)\rho(x')$. If $u \geq 0$, monotonicity gives $\log(1 - u) \leq \log(1 - ru)$. If $u < 0$, write $q = -u \leq 1$; then

$$\log(1 + q) - \log(1 + rq) \leq \frac{(1 - r)q}{1 + rq} \leq 1 - r \leq 4\varepsilon. \quad (76)$$

For the cross kernel write $a = 1 - x$, $b = \sqrt{1 - s}$, $\theta = \arccos s$, $z = 1 - xe^{i\theta}$, $z_\varepsilon = 1 - \rho(x)\sigma(s)xe^{i\theta}$; then $|z_\varepsilon - z| \leq \varepsilon|x|(a + b)$. On $x \leq 0$, $|z| \geq 1$ and $|x|(a + b) \leq 3$; on $0 \leq x \leq 1/2$, $|z| \geq 1/2$ and $|x|(a + b) \leq 1$; on $1/2 \leq x < 1$, $|z|^2 = a^2 + 2xb^2 \geq a^2 + b^2$ and $|x|(a + b) \leq \sqrt{2}|z|$. Hence $|z_\varepsilon - z| \leq 3\varepsilon|z|$ everywhere and, in the direction needed,

$$-\log |z|^2 \leq -\log |z_\varepsilon|^2 + 2 \log(1 + 3\varepsilon) \leq -\log |z_\varepsilon|^2 + 6\varepsilon. \quad (77)$$

For each fixed finite interior configuration the damped series converge absolutely. Put

$$A_k = \langle \tau^k T_k(x) \rangle, \quad B_k = \langle \rho(x)^k x^k \rangle, \quad C_k = \langle \sigma(s)^k T_k(s) \rangle.$$

Inserting diagonals gives two exact negative-square identities,

$$\frac{\kappa}{2} \langle L_{\tau^2}(x, x') \rangle = -\frac{\kappa}{2} \log 2 - \kappa \sum_{k \geq 1} \frac{A_k^2}{k}, \quad (78)$$

$$\begin{aligned} &\frac{1}{2} \langle \log(1 - \rho(x)\rho(x')xx') \rangle + \langle L_{\sigma(s)\sigma(s')}(s, s') \rangle \\ &\quad - \langle \log |1 - \rho(x)\sigma(s)xe^{i \arccos s}|^2 \rangle \\ &= -\log 2 - \frac{1}{2} \sum_{k \geq 1} \frac{(B_k - 2C_k)^2}{k}. \end{aligned} \quad (79)$$

The B_k, C_k in the cross term are exactly those of the pure terms. Adding the constants to $c_{\kappa,n} \log 2$ gives $(-1 + \alpha + \gamma - \frac{\kappa+1}{2n}) \log 2$; the omitted-diagonal constants enter the next $O_\varepsilon(1/n)$ correction.

Control this correction before taking any supremum. A damped cosine diagonal is at least $-\log 2 + 2 \log(1-r)$, so the x cosine diagonals cost $O_\varepsilon(1/n)$. For the power diagonal, $1 - \rho(x)^2 x^2 \geq 1 - x$ ($x \geq 0$) and $1 - \rho(x)^2 x^2 \geq 1 - (1-\varepsilon)^2 \geq \varepsilon$ ($x \leq 0$); as $1 \leq 1-x \leq 2$ for $x \leq 0$, the upward correction is at most $O_\varepsilon(1/n) - \frac{1}{2n} \langle \log(1-x) \rangle$. Finally $1 - \sigma(s)^2 = \varepsilon \sqrt{1-s} (2 - \varepsilon \sqrt{1-s}) \geq \varepsilon \sqrt{1-s}$, so the s cosine diagonals cost at most $O_\varepsilon(1/n) - n^{-1} \langle \log(1-s) \rangle$. The cross term already contains all pairs, and there is no loss involving $\log|x|$ or $\log(1+x)$.

For real a, b , $-a^2 \leq b^2 - 2ab$. Apply this to the first square with $b = p_k$ and the second with $b = v_k$; dividing by k and summing,

$$\begin{aligned} -\kappa \sum_k A_k^2/k &\leq \kappa \|p\|_*^2 - 2\kappa \sum_k p_k A_k/k, \\ -\frac{1}{2} \sum_k (B_k - 2C_k)^2/k &\leq \frac{1}{2} \|v\|_*^2 - \sum_k v_k B_k/k + 2 \sum_k v_k C_k/k. \end{aligned}$$

For $\kappa = 1$, failure of (59) says $1 - 2\langle x^2/(1+x^2) \rangle > 1 - 1/n - 2\gamma$, hence $\langle D(x) \rangle > -1/n$; adding $\lambda \langle D(x) \rangle$ then costs at most λ/n for an upper bound, explaining $\lambda \geq 0$ and its sign in (70).

Let $T_\varepsilon(p, x), S_\varepsilon(v, x), T_\varepsilon(v, s)$ denote the three series of (69) with factors $\tau^k, \rho(x)^k, \sigma(s)^k$. The preceding bounds reduce the estimate to two one-variable suprema of

$$\begin{aligned} X_{n,\varepsilon}(x) &= \frac{19}{48} \log|x| + \left(\frac{1}{12} - \frac{1}{2n}\right) \log(1-x) \\ &\quad - (\kappa/2 + \alpha + \eta + \gamma) \log(1+x^2) + \lambda D(x) \\ &\quad - 2\kappa T_\varepsilon(p, x) - S_\varepsilon(v, x), \\ Y_{n,\varepsilon}(s) &= \frac{7}{48} \log s + \left(\frac{1}{12} - \frac{1}{n}\right) \log(1-s) + 2T_\varepsilon(v, s), \end{aligned}$$

plus the constant in (70) and $O(\varepsilon) + O_\varepsilon(1/n)$, all bounded finite-size terms of (72) included in this error. For $n \geq 24$ both weakened endpoint coefficients are at least $1/24$.

Set $P_1 = \sum_k |p_k|$ and $V_1 = \sum_k |v_k|$, finite. From $1 - (1-d)^k \leq kd$,

$$|T_\varepsilon(p, x) - T(p, x)| \leq \varepsilon P_1, \quad |S_\varepsilon(v, x) - S(v, x)| \leq 2\varepsilon V_1, \quad |T_\varepsilon(v, s) - T(v, s)| \leq \varepsilon V_1.$$

The changes in the two tangent functions are at most $2\kappa\varepsilon P_1 + 2\varepsilon V_1$ and $2\varepsilon V_1$, common summable bounds; no uniform convergence of the raw square series is needed. All remaining nonsingular terms are uniformly bounded for $n \geq 24$ and $0 < \varepsilon < 1/8$. The positive coefficients $19/48, 7/48$ and the weakened coefficients at least $1/24$ force uniform decay to $-\infty$ near $x = 0, x = 1, s = 0, s = 1$; the values at $x = -1/2, s = 1/2$ have a common finite lower bound, and the regular endpoint $x = -1$ is included by continuous extension. Hence compact sets independent of $n \geq 24$ and small ε contain maximizers, on which the functions converge uniformly as $n \rightarrow \infty$ with ε fixed and then as $\varepsilon \downarrow 0$. Taking limits in this order removes $O_\varepsilon(1/n)$ first and proves (70); applying (67) finishes. $\square$

7 An exact certificate for the two barriers

All finite decimals in this Section are exact rationals. We specify trial sequences for (70), isolate every stationary point of its two one-variable functions, and bound their values by rational arithmetic.

Proposition 7.1. *For the determinants of (7),*

$$\limsup_{N \rightarrow \infty} \left(\frac{\log |\Delta_N|}{n^2} - \frac{1}{2} \log 2 \right) \leq -2.290939875 < -2.2909. \quad (80)$$

The separate bounds for the majorants in cases $\kappa = 2$ and $\kappa = 1$ are -2.290939875 and -2.296789875 .

7.1 Exact trial sequences

For $\kappa = 2$ take $d = 10$, $\lambda_2 = 0$; for $\kappa = 1$ take $d = 8$, $\lambda_1 = 2.47405979$. Write each sequence as

$$u_k = l_k + \sum_z r_z z^k, \quad l_k = 0 \ (k > d). \quad (81)$$

The tables give all coefficients; every integer in them is multiplied by 10^{-8} . Finite parts, in increasing k :

κ	u	finite coefficients				
2	p	45559127	-50750856	-6578767	13970217	4786184
		-4292433	-576704	1311615	671564	-346453
2	v	-23910158	21152432	-2885110	-11558199	6485289
		1456821	-1912176	-2263524	2742210	-1162454

κ	u	finite coefficients			
1	p	11913521	-79993701	-21956443	37903579
		10744022	-2073193	570246	-24939103
1	v	-89913025	52874280	45168341	-30708629
		-19269841	33922112	9819141	-16992389

Case $\kappa = 1$ has no exponential terms. For $\kappa = 2$, a real base z has coefficient a of z^k ; a nonreal base has a, b the coefficients of $\Re(z^k), \Im(z^k)$, represented in (81) by $r_z = (a - ib)/2$, $r_{\bar{z}} = \bar{r}_z$ (again $\times 10^{-8}$), each nonreal base paired with its conjugate:

u	base z	a	b
p	.85	-9338452	—
	.94	-2141509	—
	.7i	-66277922	-31907569
	.85i	-1231651	6002645
	.092 + .92i	-3225918	8928234
	-.092 + .92i	-2105536	-9091287
	—	—	—
v	-.8	15199211	—
	-.96	4451662	—
	.88	2545398	—
	.95	-4932634	—
	.984	11618157	—
	.78i	27238714	-38447936
	.9i	-31341084	-30188786
	.955i	-6693542	11912254
	.984i	2055213	-21715849
	—	—	—

These are real exponentially decaying sequences: ten p tail terms and thirteen v tail terms counting conjugates, with base moduli at most .94 and .984, and all $|l_k|, |r_z| < 1$. The convergent logarithm series give

$$\begin{aligned}\|u\|_*^2 &= \sum_{k=1}^d \frac{l_k^2 + 2l_k \sum_z r_z z^k}{k} - \sum_{z, z'} r_z r_{z'} \text{Log}(1 - zz'), \\ T(u, x) &= \sum_{k=1}^d \frac{l_k T_k(x)}{k} - \frac{1}{2} \sum_z r_z \text{Log}(1 - 2xz + z^2), \\ S(u, x) &= \sum_{k=1}^d \frac{l_k x^k}{k} - \sum_z r_z \text{Log}(1 - xz).\end{aligned}\tag{82}$$

Here Log is the principal logarithm, all sums real by conjugation, and there is no branch ambiguity. For the norm and power terms the arguments have positive real part; for the cosine term, $1 - 2xz + z^2 = (1 - ze^{i\theta})(1 - ze^{-i\theta})$ with $x = \cos \theta$ has both factors of positive real part and principal arguments summing strictly inside $(-\pi, \pi)$, so the sum of their logarithms is exactly the principal logarithm of the product, even when the product is negative real.

For these choices put

$$\begin{aligned}X_\kappa(x) &= W_\kappa(x) + \lambda_\kappa D(x) - 2\kappa T(p, x) - S(v, x), \\ Y_\kappa(x) &= \beta \log x + \gamma \log(1 - x) + 2T(v, x).\end{aligned}\tag{83}$$

The domain of X_κ is $[-1, 1) \setminus \{0\}$ and that of Y_κ is $(0, 1)$.

7.2 Derivative numerators and exhaustive root brackets

Define

$$\begin{aligned}t_u(x) &= \sum_{k=1}^d l_k U_{k-1}(x) + \sum_z \frac{r_z z}{1 - 2xz + z^2}, \\ h_u(x) &= \sum_{k=1}^d l_k x^{k-1} + \sum_z \frac{r_z z}{1 - xz}.\end{aligned}$$

Differentiating (82),

$$\begin{aligned}X'_\kappa(x) &= \frac{\alpha + 2\gamma}{x} - \frac{2\eta}{1 - x} - (\kappa + 2\alpha + 2\eta + 2\gamma) \frac{x}{1 + x^2} \\ &\quad - \frac{4\lambda_\kappa x}{(1 + x^2)^2} - 2\kappa t_p(x) - h_v(x), \\ Y'_\kappa(x) &= \frac{\beta}{x} - \frac{\gamma}{1 - x} + 2t_v(x).\end{aligned}\tag{84}$$

For completely specified rational numerators use

$$\begin{aligned}Q_X(x) &= x(1 - x)(1 + x^2)^{3-\kappa} \prod_{z \text{ in } p} (1 - 2xz + z^2) \prod_{z \text{ in } v} (1 - xz), \\ Q_Y(x) &= x(1 - x) \prod_{z \text{ in } v} (1 - 2xz + z^2), \\ A_X(x) &= Q_X(x) X'_\kappa(x), \quad A_Y(x) = Q_Y(x) Y'_\kappa(x).\end{aligned}\tag{85}$$

Empty products equal 1. These equations, the coefficient tables, and $T_0 = 1, T_1 = x, U_0 = 1, U_1 = 2x, V_{k+1} = 2xV_k - V_{k-1}$, specify four polynomials over $\mathbb{Q}$ without numerical root calculation. The denominators do not vanish on the stated domains: $|1 - xz| \geq 1 - |z| > 0$ and $|1 - 2xz + z^2| \geq (1 - |z|)^2 > 0$ for real $|x| \leq 1$; conjugate factors in Q_X, Q_Y have positive products, so Q_X has the sign of x on its domain and $Q_Y > 0$ on $(0, 1)$.

Here is an exact root-count certificate. If $A(x) = \sum_{j=0}^{d_0} A_j x^j$, on a subinterval (b, c) form

$$(1+t)^{d_0} A\left(\frac{b+ct}{1+t}\right) = \sum_{h=0}^{d_0} a_h t^h, \quad a_h = \sum_{j=0}^{d_0} \sum_{k=0}^j A_j \binom{j}{k} b^{j-k} c^k \binom{d_0-j}{h-k}, \quad (86)$$

out-of-range binomials zero. Delete zero coefficients and count consecutive sign changes. Degrees and counts, per consecutive interval:

κ	function	d_0	division points	sign variations
2	X	36	-1, 0, 1	9, 9
2	Y	24	0, .25, .5, .75, 1	5, 2, 2, 6
1	X	13	-1, -.5, 0, .5, 1	1, 1, 2, 1
1	Y	9	0, 1	5

The bound is Descartes' rule of signs [7, Book III, p. 373]. For completeness it follows by factoring positive roots: multiplication by $t - r$, $r > 0$, raises the variation count by at least one. Rescale to $r = 1$ and drop zero leading coefficients; if $a_0, \dots, a_d$ are the old and $b_0, \dots, b_{d+1}$ the new coefficients, then $\sum_{j=0}^h b_j = -a_h$ ($0 \leq h \leq d$) and $b_{d+1} = a_d$. Each sign change of the nonzero partial sums forces an intervening b_j of the later sign; starting from $b_0 = -a_0$ this gives an ordered subsequence with all variations of the a_h , and b_{d+1} has the opposite sign to the last partial sum, adding one more. So the count bounds the positive roots with multiplicity. The substitution sends $t > 0$ bijectively to (b, c) and preserves multiplicities.

The integers m below specify disjoint open brackets $(m, m+2)/10^{10}$ for roots of the corresponding derivative:

κ	function	m		
2	X	-9601109148	-8942317572	-7608305633
		-6503394794	-5185864065	-4015634158
		-3108806646	-2067921826	-1589849496
		1531948062	2072448179	3208186484
		4381119427	5851354199	7269030693
		8390277402	9332614564	9709786219
2	Y	176402802	330649406	764952882
		1227250753	2149465998	3048189112
		4322699096	5564757994	6801929373
		8031988371	8877037851	9577761832
		9838463999	9972727815	9992037196
1	X	-9917299785	-2259572153	2543808026
		4437270259	6348298970	
1	Y	532669786	2504239325	5701738806
		7966939383	9454490138	

For each integer the exact sign check is

$$N_A(m)N_A(m+2) < 0, \quad N_A(s) = \sum_{j=0}^{d_0} A_j(10^{10})^{d_0-j} s^j. \quad (87)$$

Equations (85), (86) and (87) give rational recipes for every entry. The bracket count in each interval equals the displayed variation count, so by the intermediate value theorem and the variation bound there is exactly one simple root per bracket and no other roots in those intervals. Roots at division points are harmless because all finite division points are evaluated separately below.

7.3 Rational logarithms and all candidate values

For a positive rational $s = 2^m y$ with $1 \leq y \leq 2$,

$$H(y) = 2 \sum_{j=0}^{17} \frac{1}{2j+1} \left(\frac{y-1}{y+1} \right)^{2j+1}, \quad (88)$$

with $\log s$ replaced by $mH(2) + H(y)$. If $q = (y-1)/(y+1)$, the positive remainder obeys $0 \leq \log y - H(y) \leq 2q^{37}/(37(1-q^2))$. For a nonzero rational complex $a+ib$, the real log part is half the real log of a^2+b^2 ; for the principal argument choose an octant: rotate by $-k\pi/4$, $-4 \leq k \leq 4$, so the rotated real part is positive and the ratio t satisfies $|t| \leq 1/2$, and select the rotation with $k\pi/4 + \arctan t$ principal (value π on the negative real axis). The ratio t is rational, since the rotation is, up to positive scale, a product of maps $(a, b) \mapsto (a+b, b-a)$ or $(a-b, a+b)$. With

$$J(t) = \sum_{j=0}^{23} \frac{(-1)^j t^{2j+1}}{2j+1}, \quad (89)$$

substitute $k(J(1/2) + J(1/3)) + J(t)$ for the argument; the identity $\arctan(1/2) + \arctan(1/3) = \pi/4$ follows from the tangent addition formula with both summands positive summing below $\pi/2$, and $|\arctan t - J(t)| \leq |t|^{49}/49$.

All positive inputs lie between 2^{-100} and 2^{100} : real arguments at evaluation points are between .0007 and 2, and squared moduli of complex arguments between $(.016)^4$ and $(1+.984)^4$ by the factor bounds after (85); at most 100 scaling steps suffice. With $h_0 = 2(1/3)^{37}/(37 \cdot 8/9)$ and $j_0 = (1/2)^{49}/49$, a real log costs at most $101h_0$ and an argument at most $9j_0$, and $2 \cdot 101h_0 + 9j_0 < 7 \cdot 10^{-16}$, so 10^{-15} bounds the absolute error of a complex log (hence 10^{-11} per log). Rational polynomial terms are exact. The total log coefficient mass is at most $2 \cdot 10^2 + 13^2/2 = 284.5$ in $\kappa\|p\|_*^2 + \|v\|_*^2/2$, less than $2 \cdot 10 + 13 + 2 = 35$ in X_κ , and less than $13 + 1 = 14$ in Y_κ , giving substitution errors below $2.85 \cdot 10^{-13}$, $3.5 \cdot 10^{-14}$, $1.4 \cdot 10^{-14}$, all under 10^{-8} . Enclosing each series term between multiples of 10^{-40} , adding remainder bounds with outward signs and propagating intervals linearly adds under 10^{-32} ; for complex weight c use $\Re(c \operatorname{Log} z) = \Re(c)\Re \operatorname{Log} z - \Im(c)\Im \operatorname{Log} z$. This is a fully rational interval calculation with no numerical logarithms.

Table 1 gives all fifty point-value upper bounds; its argument column is $10^{10}x$, a row marked B is a left bracket endpoint, a row marked P is a finite division point, and each bound is a rational upper bound rounded up at the twelfth decimal after the remainder is included. The largest unrounded interval width is below $2.16146856249281 \cdot 10^{-16}$.

Table 1: Certified rational upper bounds at every candidate evaluation point.

κ	function	$10^{10}x$	type	upper bound
2	X	-9601109148	B	-0.984034048775
2	X	-8942317572	B	-0.984375363807
2	X	-7608305633	B	-0.984034052640
2	X	-6503394794	B	-0.984105522615
2	X	-5185864065	B	-0.984034053414
2	X	-4015634158	B	-0.984092061375
2	X	-3108806646	B	-0.984034037901
2	X	-2067921826	B	-0.984364031075
2	X	-1589849496	B	-0.984034038450
2	X	1531948062	B	-0.984033385315
2	X	2072448179	B	-0.984776982913
2	X	3208186484	B	-0.984034026898
2	X	4381119427	B	-0.984264010107
2	X	5851354199	B	-0.984034029391
2	X	7269030693	B	-0.984245044505
2	X	8390277402	B	-0.984034016294
2	X	9332614564	B	-0.984567630027
2	X	9709786219	B	-0.984033926884
2	X	-10000000000	P	-0.986727371546
2	Y	176402802	B	-1.608946411646
2	Y	330649406	B	-1.609949505120
2	Y	764952882	B	-1.608960295828
2	Y	1227250753	B	-1.609094597854
2	Y	2149465998	B	-1.608960426500
2	Y	3048189112	B	-1.608992193672
2	Y	4322699096	B	-1.608960428486
2	Y	5564757994	B	-1.608979509517
2	Y	6801929373	B	-1.608960430478
2	Y	8031988371	B	-1.608990548812
2	Y	8877037851	B	-1.608960429811
2	Y	9577761832	B	-1.609055802895
2	Y	9838463999	B	-1.608960412835
2	Y	9972727815	B	-1.609694899071
2	Y	9992037196	B	-1.608958123669
2	Y	2500000000	P	-1.608973617030
2	Y	5000000000	P	-1.608971701898
2	Y	7500000000	P	-1.608976908052
1	X	-9917299785	B	-2.778491531574
1	X	-2259572153	B	-1.324666731948
1	X	2543808026	B	-1.324655807046
1	X	4437270259	B	-1.352236629957
1	X	6348298970	B	-1.324666329425
1	X	-10000000000	P	-2.775818077526

Table 1 (continued)

κ	function	$10^{10}x$	type	upper bound
1	X	-5000000000	P	-1.544057819905
1	X	5000000000	P	-1.347557680876
1	Y	532669786	B	-1.428286151250
1	Y	2504239325	B	-1.515602647362
1	Y	5701738806	B	-1.428335732372
1	Y	7966939383	B	-1.465686164672
1	Y	9454490138	B	-1.428335358167

The same calculation gives upper bounds .778415976284 ($\kappa = 2$) and .931985203901 ($\kappa = 1$) for $\kappa\|p\|_*^2 + \frac{1}{2}\|v\|_*^2$, so the substituted expressions, before error allowances, obey the stronger cutoffs

κ	$\kappa\ p\ _*^2 + \ v\ _*^2/2$	X	Y
2	.77843	-.98400	-1.60891
1	.93205	-1.32442	-1.42805

each a strict upper bound. With the 10^{-8} allowance the coarser cutoffs we use are

κ	$\kappa\ p\ _*^2 + \frac{1}{2}\ v\ _*^2$	X	Y
2	.77844	-.98399	-1.60890
1	.9321	-1.3244	-1.4280.

(90)

The procedure also gives $.693146 < \log 2 < .693149$.

7.4 From bracket values to global suprema

A single derivative bound suffices on every whole closed bracket. All its points are at distance greater than .0007 from both 0 and 1; the bracket nearest $s = 1$ has right endpoint $9992037198/10^{10}$, distance exactly .0007962802 from 1. For $|x| \leq 1$, $U_{k-1}(\cos \theta) = \sin(k\theta)/\sin \theta$ gives $|U_{k-1}(x)| \leq k$, including endpoint limits. The polynomial part of X'_κ is at most $4 \sum_{k=1}^{10} k + 10 = 230$; its endpoint terms are bounded by $(19/48 + 1/12)/.0007 < 685$; its $x/(1+x^2)$ term is below 3; its multiplier term below $4\lambda_1 < 10$; and its two log-tail derivatives by $4 \cdot 10/ (.06)^2 + 13/.016 < 11924$. Hence $|X'_\kappa| < 12852 < 120000$ throughout every bracket. For Y'_κ the polynomial part is at most 110, the endpoint terms below $(7/48 + 1/12)/.0007 < 328$, and the log tails below $26/ (.016)^2 = 101562.5$, so $|Y'_\kappa| < 102001 < 120000$ on every whole bracket. These hold in both cases, with empty tails in case 1.

By the mean value theorem, moving from a left bracket endpoint to its stationary point raises either value by less than $120000 \cdot 2/10^{10} = .000024$. Every interior stationary point lies in one of these brackets or is a division point; all finite division points are in Table 1 (-1 for X_2 ; $1/4, 1/2, 3/4$ for Y_2 ; $-1, -1/2, 1/2$ for X_1). The remaining endpoints are singular with limit $-\infty$: $x \rightarrow 0^\pm$ and $x \rightarrow 1^-$ for X_κ , $x \rightarrow 0^+$ and $x \rightarrow 1^-$ for Y_κ , unaffected by the bounded tangent and multiplier terms. So the root exhaustion, the finite endpoint values and the derivative estimate certify the global suprema, not just the sampled values.

Finally $-1 + \alpha + \gamma = -11/16$. Apply Proposition 6.1, use (90), and add .000024 per supremum.

Since $\log 2 > .693146$, the right side is strictly less than

$$\begin{aligned} -\frac{11}{16}(.693146) + .77844 - .98399 - 1.60890 + .000048 &= -2.290939875, \\ -\frac{11}{16}(.693146) + .9321 - 1.3244 - 1.4280 + .000048 &= -2.296789875. \end{aligned}$$

Taking the larger proves Proposition 7.1.

8 Conclusion

Proof of Theorem 1.1. Suppose G is rational. Proposition 2.2 makes Δ_N rational for every N , and Proposition 4.1 gives $\Delta_p \neq 0$ for every sufficiently large prime p . Along this prime sequence, Proposition 3.4 gives

$$\liminf_{\substack{p \rightarrow \infty \\ p \text{ prime}}} \left(\frac{\log |\Delta_p|}{(48p)^2} - \frac{1}{2} \log 2 \right) > -2.29084,$$

while Proposition 7.1 gives, along the same sequence, an upper limit strictly below -2.2909 . Since $-2.29084 > -2.2909$, this is a contradiction. Hence G is irrational. $\square$

Hyperbolic volumes. Normalize sectional curvature to -1 . Agol's theorem identifies $4G$ as the minimum volume of an orientable complete finite-volume hyperbolic three-manifold with exactly two cusps, attained by the Whitehead-link and $(-2, 3, 8)$ -pretzel-link complements [1, Introduction and Theorem 3.6]. Theorem 1.1 therefore makes this minimum and both link volumes irrational. More generally, every orientable arithmetic hyperbolic three-orbifold defined over $\mathbb{Q}(i)$ has irrational volume. Its lattice $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ is, up to conjugacy, commensurable with the projective norm-one group $\Gamma^1(\mathcal{O}) = \mathcal{O}^\times / \{\pm 1\}$ of a maximal order $\mathcal{O}$ in a quaternion algebra $B/\mathbb{Q}(i)$ [24, Definition 38.3.4]. For each such algebra with finite reduced discriminant $\mathfrak{D}$, the volume formula gives

$$\mathrm{vol}(\Gamma^1(\mathcal{O}) \backslash \mathbb{H}^3) = \frac{G}{3} \prod_{\mathfrak{p} | \mathfrak{D}} (N\mathfrak{p} - 1), \quad N\mathfrak{p} = |\mathbb{Z}[i]/\mathfrak{p}|,$$

obtained by inserting $\zeta_{\mathbb{Q}(i)}(2) = \zeta(2)L(2, \chi_{-4}) = (\pi^2/6)G$ into [24, Theorem 39.1.13]. After conjugating, a common finite-index subgroup H gives

$$\mathrm{vol}(\Gamma \backslash \mathbb{H}^3) = \frac{[\Gamma^1(\mathcal{O}) : H]}{[\Gamma : H]} \mathrm{vol}(\Gamma^1(\mathcal{O}) \backslash \mathbb{H}^3),$$

a positive rational multiple of G , hence irrational. In the split case $B = M_2(\mathbb{Q}(i))$, $\mathcal{O} = M_2(\mathbb{Z}[i])$, the empty product yields $\mathrm{vol}(\mathrm{PSL}_2(\mathbb{Z}[i]) \backslash \mathbb{H}^3) = G/3$ [24, Example 39.1.16].

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