

Bounded scalar curvature and smooth extension of four-dimensional Ricci flow

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Abstract

We prove that a smooth Ricci flow on a closed real four-manifold extends on the same manifold through every finite time at which its scalar curvature stays uniformly bounded: at a finite-time singularity the scalar curvature must blow up. The proof combines a scalar-curvature compactness and bubbling description of the degenerating slices with a renormalized Einstein–Hilbert functional and a single monotone quantity, the spacetime L^2 norm of the Ricci tensor in the outermost bubble.

Strategy of the proof.

The proof follows one strategy. Fix a hypothetical concentration point where curvature blows up at T . Successive rescalings produce a finite tree of Ricci-flat limits; the outermost nonflat limit defines a scale $a(t)$ that cannot change by any fixed factor on short intervals. On the corresponding buffered intervals, parabolic rescaling places the Ricci energy K of the outermost bubble in a limiting spacetime where the renormalized Einstein–Hilbert functional and the companion tree inequality force $K \rightarrow 0$; that pins $a(t)$ and rules out the concentration point. The mechanics are in four blocks. Block I (“The spatial curvature radius” to “Buffered intervals”) estimates curvature at all scales, describes the slice limits, and selects the buffered intervals. Block II (“Concentration estimates” to “The maximal function”) bounds the Ricci heat equation and the root energy. Block III (“Collar extension”) builds and estimates the analytic exterior from a fixed collar. Block IV (“Functional change and the contradiction” to “Continuation”) supplies the lower and upper bounds on the functional variation and closes the argument. A reader focused on the main idea should read Blocks I and IV first, treating Block III as a black box for the error analysis.

Introduction.

A Ricci flow is a one-parameter family of Riemannian metrics solving the heat-type equation

$$\partial_t g = -2 \operatorname{Ric}(g). \quad (1)$$

On a closed manifold it exists and is smooth for a short time; the central question is what makes the maximal smooth existence time $T < \infty$ finite. The full curvature tensor decides this: if $|\operatorname{Rm}|$ stays bounded on $[0, T)$, the flow continues past T [1]. The scalar curvature $R = \operatorname{tr}_g \operatorname{Ric}$ is only a trace, so a bound on R is far weaker; it controls volume by $\partial_t d\mu_g = -R d\mu_g$ but leaves the full geometry free. The scalar-curvature extension problem asks whether a uniform bound on R alone already forbids a finite-time singularity. The answer is affirmative in real dimension four.

Theorem 1 (Smooth extension). *Let M be a closed connected smooth four-manifold, $0 < T < \infty$, and let $g(t)$, $0 \leq t < T$, be a smooth Ricci flow from smooth initial metric with $\sup_{M \times [0, T)} |R(g(t))| < \infty$. Then there are $\varepsilon > 0$ and a smooth Ricci flow $\tilde{g}$ on M for $0 \leq t < T + \varepsilon$ with $\tilde{g}(t) = g(t)$ for all $t < T$.*

Corollary 2. *Let $g(t)$, $0 \leq t < T < \infty$, be a smooth Ricci flow on a closed four-manifold with finite maximal smooth existence time T . Then $\limsup_{t \uparrow T} \max_M R(g(t)) = +\infty$.*

Proof. $(\partial_t - \Delta)R = 2|\text{Ric}|^2 \geq 0$ and the maximum principle bound R below on any finite interval. A finite upper limit would bound $|R|$ near T ; since R is smooth on the compact initial part, Theorem 1 applied componentwise would contradict maximality of T . $\square$

The proof rules out contraction of a specific outermost bubble scale. A concentration point produces a finite rooted tree of Ricci-flat limits, one for each scale at which curvature blows up; the outermost nonflat limit has a length $a(t)$ that is positive before T but tends to zero at the singular time. On short intervals a cannot change by a fixed factor: rescaling would freeze the pointed geometry, so any fixed-factor change would contradict the identity of a fixed limiting tree. Parabolic rescaling on the buffered intervals therefore produces a limiting spacetime in which the Ricci energy K of the outermost bubble is comparable from start to end. The renormalized Einstein–Hilbert functional computed on that scaling, together with a sharp inequality from the companion paper, forces $K \rightarrow 0$. But $K \rightarrow 0$ means distances are preserved on the concentration scale, which pins $a(t)$ and contradicts its prescribed finite change. Hence no concentration point exists and the flow extends smoothly.

The geometric obstruction.

Hamilton [1] proved short-time existence; Šešum [2] showed a uniform bound on Ric forces continuation. A scalar bound is weaker because after rescaling a would-be singularity the scalar curvature can tend to zero while full curvature stays bounded away from zero. The limit is then Ricci-flat without being flat, so a scalar bound permits a whole tree of collapsing Ricci-flat bubbles.

Bamler–Zhang [3, 4] proved the heat-kernel and curvature estimates under scalar bounds, including the four-dimensional energy and compactness theory used below, and obtained smooth extension when $H_2(M; \mathbb{F}) = 0$ for all $\mathbb{F}$ [3, Corollary 1.10]. Simon [5] continued through the limiting space by an orbifold flow; Bamler’s convergence theory [6] gives smooth convergence off a codimension-four set; Liu–Simon [7] treat four-dimensional convergence under scalar-integrability hypotheses. All allow concentrated curvature and orbifold points; the continuation below instead keeps the given flow on its original manifold. Blow-up is known for Kähler–Ricci flows [8], for global Type I [9], and for local Type I points [10]; no rate is imposed.

The spatial picture descends from Einstein four-manifold theory: Anderson [11], Bando [12, 13], Bando–Kasue–Nakajima [14], Cheeger–Tian [15]. The slices below are not Einstein, so the needed statements are reproved under a scalar bound, retaining the Ricci forcing.

The outermost scale and the static functional.

Successive rescalings at a concentration point produce a finite rooted tree of Ricci-flat pieces; the outermost nonflat vertex has a length scale $a(t)$ described below. The proof rules out contraction of $a(t)$ while allowing arbitrarily small descendants.

Retain the true metric through an annular collar of radius $b = Na$ (N large, fixed) and attach an ALE end in Euclidean quotient coordinates. The renormalized Einstein–Hilbert functional is

$$E(g) = \lim_{D \rightarrow \infty} \left[\int_{\{r < D\}} R d\mu_g - \int_{\{r=D\}} (\partial_j g_{ij} - \partial_i g_{jj}) n_i dS_\delta \right],$$

the mass term of Deruelle–Ozuch [22, Definition 4.2, Proposition 4.4]. It scales by the square of the length unit and its first variation pairs the velocity with $\frac{1}{2}Rg - \text{Ric}$. The companion article [26] proves the sequential inequality $E = o(M)$ on each fixed limiting tree (M the weighted,

dimensionless Ricci error). The static input is recorded below; the argument must verify the hypotheses for flow metrics and compare the functional variation with spacetime Ricci energy.

Two estimates drive the argument. On a nearly flat annulus the contracted Bianchi identity removes the critical radial harmonic mode of Ric, improving the annular decay beyond the dimensional r^{-2} rate so the loss outside the collar is small against the energy inside. The exterior beyond the collar is built as an analytic function of the metric on the one fixed material collar, with the longer exterior entering only an error estimate; hence the extension differentiates in time on patches without a smoothly varying tree. This rests on the Saint-Venant relation [23] and Korn control modulo rigid motions [24], both used in the collar construction below.

Proof structure and conventions.

The static input is fixed first. The scale estimates, genuine orbifold limits, finite trees, and the continuous scale $a(t)$ follow, together with buffered intervals on which a changes by a fixed factor with root length tending to zero after parabolic normalization. The Ricci-equation estimates come next: the exterior decay beyond the outermost scale, and the maximal and weighted interior estimates controlling the root energy. The collar construction then builds the exterior extension and its error. A final comparison of functional variation with Ricci energy: an explicit lower bound cK^2 and upper bound $\zeta_i(K+1)^2$ force $K \rightarrow 0$; then distances are preserved so the curvature-radius maximum defining a is preserved, contradicting its prescribed change, while absence of a concentration point gives a smooth terminal metric and continuation on M .

Throughout, $\Delta = \text{tr } \nabla^2$ and $|A|_{k,r} = \sum_{j=0}^k r^j |\nabla^j A|$ (the pointwise magnitude unchanged, derivatives scaled). Quotient-chart components are Cartesian components on the Euclidean cover; constructions are equivariant and descend. Constants may depend on a fixed finite differentiation order and, after extraction, on a fixed finite tree; uniformity in a sequence index, length or collar factor N is stated where needed, and a sequential little-oh is taken after fixing the parameters in its statement.

The static input and its normalization.

The input from [26, Theorem 2.2] is recorded here. Constants and small exponents belong to one fixed limiting tree; no uniform exponent across trees is needed.

Geometric data and the functional.

Let $\mathcal{T}$ be a finite rooted tree of four-dimensional Ricci-flat spaces (V_v, h_v) . A vertex has one exterior end and one punctured end per child. Every end carries Cartesian coordinates on a Euclidean annulus modulo a finite subgroup of $O(4)$ acting freely on S^3 ; the group on a joined end is nontrivial, and the two groups on an edge are identified with a fixed orthogonal identification included in the charts. Put $\tau = \log r$ on an exterior end and $\tau = -\log r$ at a puncture. In vertex units,

$$h_v = \delta + O(e^{-\mu\tau}) \quad (2)$$

on joined ends, with scaled derivatives, for some $\mu > 0$; the unjoined exterior end of the root requires $h_v - \delta = O(r^{-q})$ with $q > 1$. Bounds through order 20 suffice.

For an edge from a parent p to a child c truncate both end coordinates at $\tau = L_e$ and identify the collars by

$$x_p = e^{-2L_e} x_c, \quad a_c = a_p e^{-2L_e}, \quad a_{\text{root}} = 1. \quad (3)$$

The physical radius $\rho = a_v r$ agrees in both charts and $\tau_p + \tau_c = 2L_e$; the metric in vertex units is multiplied by a_v^2 in physical units. Gluing the boundary faces with their compatible collars produces the manifold.

Assume the sequence of metrics has all $L_e \rightarrow \infty$ and converges to h_v on compacts of every open vertex, with uniform weighted Ricci control

$$|\rho^2 \text{Ric}|_j \leq CM \begin{cases} e^{-\mu\tau}, & \text{joined half-ends} \\ r^{-q}, & \text{root exterior} \\ 1, & \text{cores,} \end{cases} \quad j = 10, \quad (4)$$

with $M \rightarrow 0$ (tensor magnitude physical; covariant derivatives scaled by powers of ρ). Decreasing μ or q is allowed.

Define, in physical root units,

$$E(g) = \lim_{D \rightarrow \infty} \left\{ \int_{r < D} R d\mu_g - \int_{r=D} (\partial_j g_{ij} - \partial_i g_{jj}) n_i dS_\delta \right\}. \quad (5)$$

All chart integrals are quotient integrals.

Lemma 3 (Renormalization and first variation). *The functional (5) is defined on the preceding metrics. For a variation with the same root decay,*

$$DE_g(\dot{g}) = \int \left\langle \frac{1}{2} Rg - \text{Ric}, \dot{g} \right\rangle_g d\mu_g. \quad (6)$$

It is invariant under diffeomorphisms isotopic to the identity whose root displacement, with scaled derivatives along the isotopy, is $O(r^{1-q'})$, $q' > 1$. A global scaling by s , with a coordinate dilation restoring δ , multiplies E by s^2 . Under (4), $|E| \leq CM$.

Proof. On the root end write $g = \delta + p$. The density obeys $R\sqrt{\det g} = \partial_i(\partial_j p_{ij} - \partial_i p_{jj}) + O(|p||\partial^2 p| + |\partial p|^2)$, the error $O(r^{-2q-2})$, radially integrable; this proves convergence. The linear boundary term in the variation cancels the variation of the subtracted integral, and remaining boundary products are $O(D^{2-2q}) \rightarrow 0$; integration by parts gives (6). For a Lie derivative the contracted Bianchi identity kills the interior divergence and the boundary term vanishes by the displacement decay, so the functional is invariant along the isotopy. Differentiating the scaling law at $s = 1$ with a compensating dilation (identity in a fixed outer region, metric varied by $2g$ inside and $O(r^{-q})$ on the tail, paired with an integrable Ric) gives $2|E| \leq CM$, hence $E = 0$ for Ric = 0. $\square$

Theorem 4 (Static tree inequality). *This is [26, Theorem 2.2]. For the fixed tree and sequences above,*

$$E(g) = o(M), \quad (7)$$

and for $M = 0$ it means $E(g) = 0$.

Small-scale geometry and the outermost scale.

Throughout, $(M^4, g(t))$, $0 \leq t < T$, is the closed flow with $|R| \leq C_R$. Constants may depend on the flow, T , and a fixed differentiation order.

The spatial curvature radius.

Definition 5. For a small cap $l_0 > 0$ and a material point x , $l(x, t) = \sup\{0 < r \leq l_0 : \sup_{B_t(x, r)} |\text{Rm}| \leq r^{-2}\}$.

The definition uses the whole spatial ball. Then l is positive, continuous before T , and 1-Lipschitz: $|l(x, t) - l(y, t)| \leq d_t(x, y)$ (a ball of radius $r - d_t(x, y)$ at y lies inside the radius- r ball at x with a larger allowed bound).

Theorem 6 (Scale estimates and degeneration). *For a small cap and all sufficiently late times:*

$$cr^4 \leq \text{Vol}_t B_t(x, r) \leq Cr^4, \quad \int_M |\text{Rm}|^2 d\mu_t \leq C, \quad (8)$$

$$|\nabla^k \text{Rm}|(x, t) \leq C_k l^{-2-k}, \quad |\nabla^k \text{Ric}|(x, t) \leq C_k l^{-1-k}. \quad (9)$$

The set $\{l < r\}$ is covered by a bounded number of radius- Cr balls. Distances converge uniformly to a compact metric quotient Z ; off finitely many concentration points the metric converges smoothly. Small-scale limits are complete noncollapsed Ricci-flat orbifolds, smooth off finitely many genuine orbifold points with nontrivial finite group acting freely on S^3 ; a nonflat limit has one Euclidean quotient end with nontrivial group, a flat one is a global Euclidean quotient with at most one exceptional point. Within each cluster the maximal inequivalent nonflat limits form a finite rooted tree with a unique outermost entry, and the joining regions carry quantitative neck charts.

The proof occupies this section.

Heat localization and two-sided persistence of the radius.

Fix the terminal time in the smooth part and use h -dependent conjugate heat kernels of the fixed metric, then localize them on radius- h balls by a cutoff χ equal to one on $B_t(x, h)$ and supported in $B_t(x, 4h)$ with $|\nabla\chi|$ and $h|\partial\chi|$ bounded. The localized flow has bounded curvature Ch^{-2} and bounded first spatial derivatives, so its kernel obeys the Gaussian upper and lower bounds and the derivative estimates on the ball; a conjugate heat function equals a positive multiple of this localized solution, and continuity plus the terminal value at the pole give

$$K \leq Ch^{-n/2}, \quad K \geq c'(s)h^{-n/2} \text{ on } B_s(x, h), \quad s < h. \quad (10)$$

The double integral bounds the backward heat flow $L^1 \rightarrow L^\infty$ on $[t - h^2, t]$ and the forward heat flow $L^\infty \rightarrow L^1$ on $[t, t + h^2]$ by $Ch^{-n/2}$; together with the L^2 contraction and duality, $\int v d\mu_t \leq C \int v d\mu_{t+h^2}$ for $v \geq 0$. For nonnegative $u_s(x) = u(s, x)$ with $|\nabla u_s|^2 \leq C_s$, taking $v = (u_s - u_t)_+$ gives

$$\int v d\mu_t \leq C(\sqrt{|t-s|} + \sqrt{|t-s|}\tilde{C}) \int |\nabla u_{s'}| d\mu_{s'}, \quad (11)$$

the form of Simon's backward uniqueness that needs only local bounds. For $t_0 < s < t$ and $u_s(y) = d_t(x, y)$, this fails unless $d_t(x, y) \geq c'd(s, y) - r$. Squaring and optimizing in $r^2 \leq cs$ gives spatial persistence

$$d_t(x, y) \geq c'd_s(x, y) \quad \text{if } d_s(x, y) \leq c''l_0. \quad (12)$$

For the effect on the curvature radius fix a coordinate energy $e(t, \cdot)$ for the covariant components of Rm in a fixed initial normal chart, with $e(0, \cdot) = \sum_{j \leq 12} |\nabla^j \text{Rm}|^2$. For a small fixed $\beta > 0$ the smoothing estimate $\sup_{B(x, r/2)} e(t) \leq C_\zeta r^{-6} \int_{B(x, r)} e(0)$ together with volume and derivative bounds gives $e(t) \geq |\text{Rm}(t)|^2$ and

$$|\text{Rm}(g(t))| \leq C' \left(\sup_{B(x, r)} |\text{Rm}_0|^2 + r^{-4} \right). \quad (13)$$

Applying this to rescaled flows gives the two-sided statement: if a ball has curvature radius $\geq r$ at time t , the earlier slice at distance $\geq r$ has curvature radius $\geq cr$, and conversely (using the backward kernel).

Energy. With C_0 so that $1 \leq R+C_0 \leq C$, the quotient evolution gives $(\partial_t - \Delta)|\text{Ric}|^2/(R+C_0) \leq C|\text{Rm}||\text{Ric}|^2 - c|\text{Ric}|^4$; the volume derivative and Young's inequality give, for $F = \int |\text{Ric}|^2/(R+C_0)$, $F' \leq C \int |\text{Rm}|^2 - c' \int |\text{Ric}|^4 + CF$. Chern–Gauss–Bonnet $\int |\text{Rm}|^2 \leq C+4 \int |\text{Ric}|^2 \leq C+CF$ [17, 16] (orientation cover if needed) and Gronwall bound F ; this is (8).

Bad set and fixed energy. If $l(y) \geq r$ and $r^2|\text{Rm}|(y) \geq \varepsilon$, then the derivative bound gives $|\text{Rm}| \geq \varepsilon/(2r^2)$ on $B(y, cr)$ and $\int_{B(y, cr)} |\text{Rm}|^2 \geq c\varepsilon^2 r^4$; volume comparability upgrades this to a definite Ricci energy $e_0 > 0$ on a smaller ball. Now fix small ρ and suppose $\{l < \rho\}$ needs more than M radius- $C\rho$ balls. Take a maximal family of disjoint unit balls centered at points $x_j \in \{C\rho \leq l < \rho\}$ chosen by the primitive $x \mapsto l(x)\psi(\rho/l(x))$; the derivative bound on this primitive gives disjoint definite-energy balls, and the slice energy bound yields $M \leq Ce_0^{-1}$. Hence $\{l < \rho\}$ is covered by boundedly many radius- $C\rho$ balls, with the bound independent of ρ .

Convergence of distances and volume comparison. For $|t-s| \leq cr^2$, cover the bad set at the first time by bounded radius- Cr balls, replace visits to enlarged balls by first-to-last visits (boundedly many endpoints, total cost $O(r)$ by (12)), so that on the remaining path the curvature scale is $\geq cr$ and the length changes by a factor $1+Cr$ since $|R| \leq C/r$. Comparing both ways, $|d_t(x, y) - d_s(x, y)| \leq Cr(1+d_s+d_t)$, and taking $r \asymp \sqrt{|t-s|}$ gives uniform convergence to a pseudodistance d_T and a compact quotient Z .

The slicewise L^3 bound $\int_{B(x, h)} |\text{Ric}|^3 \leq Ch$ will also be needed. On the shell $\{r_j \leq l \leq r_{j-1}\}$, $|\text{Ric}| \leq Cr_j^{-1}$ and the bad-set measure bound established just above gives measure $\leq Cr_j^4$, contributing Cr_j . Summing the dyadic shells up to h yields $C \sum_j r_j \leq Ch$.

In polar coordinates the polar Jacobian P and $m = (\partial_r \log P - 3/r)_+$ satisfy $(m^5 P)' \leq 5|\text{Ric}|m^4 P - 7m^5 P/r - (2/3)m^6 P$, giving $\int m^6 \leq C \int |\text{Ric}|^3 \leq Ch$ and hence, since $(P/r^3)' \leq mP/r^3$,

$$\frac{\text{Vol } B(x, h)}{h^4} \leq \frac{\text{Vol } B(x, h/2)}{(h/2)^4} + Ch^{1/2}, \quad (14)$$

while polar integration including the cut measure gives $\Delta(d^2/2) \leq 4 + dm$ weakly. Volume forms at two times differ by $e^{\pm C_R|t-s|}$.

Slice limits, cones, and quantitative necks.

Rescaled slices $h_i^{-2}g(t_i)$ based at x_i with $h_i \rightarrow 0$, $t_i \uparrow T$ are precompact as length spaces (two-sided volume bounds bound the number of disjoint balls); the bad-set cover has bounded cardinality so a diagonal limit identifies a finite exceptional set S . Restricted maps are locally bi-Lipschitz, extend by continuity, and preserve the injectivity radius bound, so the limit is complete; smooth convergence off S gives a smooth Ricci-flat metric there, called regular. Ball volumes converge and are regular Riemannian volume. The polar defect vanishes and Bishop becomes exact on the regular part of the limit.

The exceptional set cannot separate positive-volume regular components: a supposed separation by S forces approximating minimizers through thin neighborhoods; the polar inequality and Fubini over the passage give

$$c\rho \text{Vol}(U) \leq C \left(\text{Vol}(N_{C\rho}(S)) + \int m \right),$$

first take the sequence limit and then $\rho \downarrow 0$ to contradict. Hence the limit is a complete orbifold, singular only where point-picking gives nontrivial concentrated curvature.

Write d for distance from a puncture (or a fixed point at infinity). Vanishing annular energy rules out $r^2|\text{Rm}| \geq \varepsilon$ in a middle annulus (energy $c\varepsilon^6$ on a ball), and interior stationary estimates give the same for every fixed scaled derivative. Then $(4 - \Delta)(d^2/2) = 0$ distributionally with $\Delta(d^2/2) = 4$ and $|\nabla^2(d^2/2)| = 4$, so $\nabla^2(d^2/2) = g$ and the end is a cone $dr^2 + r^2 g_L$, $L = S^3/\Gamma$ for

finite Γ acting freely and nontrivially (nontrivial group; two-sided volumes bound $|\Gamma|$). Fixed-ratio annuli are arbitrarily close to round quotient annuli, and overlapping annuli identify the groups.

Lemma 7 (Quantitative neck). *If a nonflat complete Ricci-flat limit with $|\text{Rm}| \leq r^{-2}$ near a marked point has $0 < \mu < \sigma < 1$ and $r_- < r_+$, then for small r_-/r_+ and large enclosing annulus, after decreasing μ, σ ,*

$$|\text{Rm}|(x) \leq r^{-2}((r/r_+)^\sigma + (r_-/r)^\sigma), \quad r_- < r < r_+, \quad (15)$$

through all fixed scaled derivatives.

Proof. Trim the boundaries by a fixed factor (otherwise a sequence of violating annuli with escaping ratios contradicts smooth convergence). If the estimate fails, choose approximate maximizers of $r^2|\text{Rm}|/w_\mu(r)$, $w_\mu(r) = (r/r_+)^\mu + (r_-/r)^\mu$, with ratios $Q_i \rightarrow \infty$ at s_i ; both boundaries escape after dilation by s_i^{-1} . Dividing the rescaled curvature by $A_i = s_i^2|\text{Rm}|(s_i) \rightarrow 0$ gives a bounded divided tensor. The second Bianchi identity and $\Delta \text{Rm} = \text{Rm} * \text{Rm} + \nabla^2 \text{Ric} *$ make the divided quadratic term vanish and the forcing vanish on fixed annuli; interior elliptic estimates yield a nonzero limiting Cartesian tensor H , harmonic and satisfying the uncontracted Bianchi identity on the flat punctured cover. Spherical harmonics (radial powers $j, -j-2$) and $\mu < 1$ leave only $H = r^{-2}C$ for constant C ; Bianchi then reads $x_i C_{jkab} + x_j C_{kiab} + x_k C_{ijab} = 0$ for all x , forcing $C_{..ab} = 0$, contradiction. Derivatives follow by interior elliptic estimates; slowly varying boundary coefficients are absorbed by decreasing μ . $\square$

Removal of punctures.

Work near a punctured orbifold point. A constant rescaling makes $g(0) = \delta$; (15) gives $|g - \delta| = O(r^\mu)$, $|Dg| = O(r^{\mu-1})$, $|D^2g| = O(r^{\mu-2})$ on the lifted ball. Harmonic coordinates solve $\partial_i(A^{ij}\partial_j y^a) = 0$ with $A^{ij} = \sqrt{\det g} g^{ij}$ and $y^a = x^a$ on ∂B_R ; coercivity gives a variational solution, and after rescaling a weak gradient estimate with $C^{0,\alpha}$ forcing $O(R^\mu)$ gives $\|Dy - I\|_\infty = O(R^\mu)$, so y is a diffeomorphism for small R . Uniqueness makes it equivariant; recentering and a constant linear normalization give $D^k y = O(r^{1+\alpha-k})$ for $k \geq 2$ and the same for the inverse. The harmonic-coordinate metric obeys $G - G(0) = O(r^\alpha)$, $DG = O(r^{\alpha-1})$, $D^2G = O(r^{\alpha-2})$.

Thus $G \in W^{2,p}$ for all $p < 4/(2-\alpha)$; take $p_0 = 2 + \alpha/2 > 2$. No distributional point defect occurs (integration by parts across ∂B_ρ has first-derivative $O(\rho^3)$ and second-derivative $O(\rho^{2+\alpha})$ terms). The harmonic-coordinate Einstein equation holds a.e.,

$$G^{ab}\partial_a\partial_b G_{ij} = Q_{ij}(G^{-1}, DG, DG), \quad (16)$$

with Q quadratic in DG [19, Lemma 4.1]. For $2 < p < 4$, $DG \in L^{4p/(4-p)}$ and elliptic regularity improves the exponent, $p_{k+1} = 2p_k/(4-p_k)$ with $1/p_k = \frac{1}{2} - 2^k(\frac{1}{2} - 1/p_0)$, reaching $p > 4$ in finitely many steps; then $G \in C^{1,\beta}$, and Schauder theory with (16) gives smoothness. This fills the orbifold point without integrability assumptions on Einstein deformations.

A trivial group is impossible: its smooth filling has near-Euclidean volume ratios, which propagate (via (14), error $O(h^{1/2})$) to a smaller complete nonflat limit at a point-picked center near the filled center; letting the start ratio go to Euclidean forces at least Euclidean ratio on fixed balls of that limit, while Bishop gives the opposite at regular centers and the radial-Laplacian/Hessian argument forces a smooth flat cone, contradiction. The end group of a nonflat limit is likewise nontrivial. A complete flat limit develops to a global Euclidean quotient with at most one tip (finiteness of the deck group from sheet counting and noncollapse; the only possible singular point is the tip).

Terminal convergence, empty set, finite trees.

For a compact set K off the finite exceptional set, nested compactness and a fixed radius r control the whole remaining interval $[t_i, T)$ by spatial persistence and a finite cover, giving fixed charts, bounded derivatives, and smooth terminal convergence (local injectivity gives one sheet). If the exceptional set is empty, finitely many such charts bound curvature globally and the DeTurck equation continues the flow past T (principal part $g^{ij}\partial_i\partial_j g$ after adding $\mathcal{L}_W g$); so only a genuine concentration point needs work.

Call two nonflat small-scale limits equivalent if their scales have bounded ratio and centers are a bounded distance apart in either scale. Each nonflat limit contains a regular energy ball (orbifold filling, end decay, and the stationary curvature bound give a ball of energy at least $e_1 > 0$). Such balls for inequivalent entries are disjoint in the slices; the total energy bounds the number by C/e_1 . Dilating to curvature maximum and ordering give an ancestor relation with a unique maximal entry; recursively this is the finite rooted tree of Theorem 6. Intermediate limits between consecutive entries are flat, so a child is a cone tip; on the flat annuli $l \geq cr$ and (9) gives $r^2|\text{Ric}|_{k,r} \leq Cr$, and Lemma 7 supplies the necks with positive decay and the matching quotient groups (end variables $\tau_p = \log(A/r)$, $\tau_c = \log(r/(Ae^{-2L}))$, $\tau_p + \tau_c = 2L$ realize exactly the static half-neck weights).

The outermost scale.

Proposition 8 (A continuous outermost scale). *Fix a small $c_* > 0$ and a bounded continuous $\psi \geq 0$ with $\psi = 0$ on $[0, c_*]$, $\psi > 0$ on $[2c_*, 1]$, and set*

$$a(t) = \sup_{y \in \mathcal{U}} l(y, t) \psi(l(y, t)^2 |\text{Rm}|(y, t)). \quad (17)$$

After shrinking $\mathcal{U}$ the supremum is attained in the interior at every late time; a is positive, continuous, and $a(t) \rightarrow 0$. For every sequence of late times, $a(t)$ and any maximizer represent the outermost nonflat entry, and every other nonflat entry has scale at most Ca .

Proof. Normalized regular energy balls detect curvature: at $\max |\text{Rm}| = 1$ the derivative bound keeps $|\text{Rm}| \geq 1/2$ on a fixed ball, and removing tip neighborhoods leaves a point at positive distance from every tip with curvature radius bounded above and below, so $l^2 |\text{Rm}| \geq 2c_*$ and l is comparable to the vertex scale. Near the punctures $l^2 |\text{Rm}| < c_*/2$ eventually (definite-energy balls would accumulate against finite energy). Hence the integrand in (17) is continuous on a fixed compact interior set with an interior maximum, positive and continuous in time; maximizing points go to the concentration point with $l \rightarrow 0$, so $a(t) \rightarrow 0$. At a maximizer $l \geq a/\|\psi\|_\infty$, $l^2 |\text{Rm}| > c_*$, and $l \asymp a$ (else detection at another point forces a larger value). Any listed nonflat limit has scale $\leq Ca$ by the same detection, and the maximizing limit shares its unique outer ancestor with the maximal list, so it is outermost. $\square$

Motion on short intervals.

Lemma 9 (Curvature radius along minimizing segments). *For any minimizing segment γ at a late time, $l(\gamma(s), t) \geq c \min(l_0, s, D - s)$. Consequently, if $d_t(x, y) \leq Ch$,*

$$|\partial_t d_t(x, y)| \leq C'(1 + \log^+(h/l(x, t)) + \log^+(h/l(y, t))). \quad (18)$$

Proof. Otherwise centered subsegments of length $2h_i$, $h_i = \sqrt{r_i L_i}$, rescale to minimizing segments with an interior point of vanishing curvature radius, which is a genuine cone vertex with link S^3/Γ of

diameter $< \pi$ (two orbits at distance π force a singleton, contradicting freeness); blowing up at the vertex gives chords contradicting minimality. The derivative bound follows from $|\partial_t d| \leq \int_\gamma |\text{Ric}| ds$, (9), and $l(\gamma(s)) \geq c(l(x) + s)$ on the halves. $\square$

Lemma 10 (Uniform short motion). *Let $h_i \rightarrow 0$ and intervals J_i of length $\delta_i = o(h_i)$ tending to T . Distances of size $O(h_i)$ change by $o(h_i)$ throughout J_i ; a point with $l \geq ch_i$ retains $l \geq c'h_i$; regular charts at this scale change by $O(\delta_i/h_i)$ with scaled derivatives.*

Proof. Dense retained material set from the summability of $\int_M \log^+(h_i/l)$ (bounded by Ch_i^4) and the volume bounds; on retained endpoints (18) integrates to $o(h_i)$ change; a first-crossing argument keeps ranges bounded. Persistence of regular charts follows from the two-sided curvature persistence and the metric-difference evolution. $\square$

Lemma 11 (Time for a fixed scale change). *If $a(t_i)/a(s_i) \rightarrow 1$ along $s_i < t_i < T$ with $\log(a(t_i)/a(s_i)) \rightarrow 0$ while $(t_i - s_i)/a(s_i)^2$ stays bounded, then the root scale cannot change by a fixed factor.*

Proof. Rescale lengths by $a(s_i)$; the duration is $o(a)$, so Lemma 10 freezes the pointed root metric and its regular curvature. The radius defining a converges on regular parts to the radius computed in that limit with tips as stops, maximizers stay in a common bounded root range, and the rescaled supremum of (17) agrees at both times, excluding a fixed-factor change. $\square$

Buffered intervals, motion, and exterior geometry.

Proposition 12 (Buffered intervals). *Fix $u > 0$ and choose $t_0 \leq t_1 \leq \dots < t_N < T$ a maximal u -separated sequence for $-\log a(t)$. Then $N \rightarrow \infty$ and $\theta := \inf_j (t_{j+1} - t_j)/(t_j - t_{j-1}) > 0$.*

Proof. Successive gaps d_j satisfy $d_j/a_j^2 \rightarrow \infty$ and $d_j \geq ce^{-2uj}$ by Lemma 11. A geometric lower bound and the maximality prevent two consecutive tiny ratios beyond a fixed factor. $\square$

On a buffered interval $I = [-\theta, 1 + \theta]d_j$ with endpoints t_j, t_{j+1} and $d_j = t_{j+1} - t_j$, in interval units $g_{\text{new}} = d_j^{-1}g$ one has $|R_{\text{new}}| \leq \epsilon_i^2$ with $\epsilon_i = \sqrt{d_i} \rightarrow 0$, $a_i = a(t_j)/\epsilon_i \rightarrow 0$ throughout I (up to a fixed factor), and the root scale at 0 and 1 differs by a fixed factor. By Lemma 10, geometry at length $h \lesssim 1$ moves negligibly over windows Δt with $\epsilon \Delta t/h \rightarrow 0$; at root scale centers move $O(a_i)$, at $a_i \ll h \lesssim 1$ by $o(h)$ (a second concentration at separation h would give a larger nonflat ancestor).

Proposition 13 (Uniform exterior geometry). *Fix a high order k_{geo} and $0 < \mu < \sigma < 1$. There are $N_{\text{geom}}, C, c_{\text{geom}} > 0$ and $R_i^{\text{out}} \rightarrow \infty$ with $\epsilon_i R_i^{\text{out}} \rightarrow 0$ such that at every time in I the exterior admits one equivariant coordinate system with*

$$r^2 |\text{Rm}|_{k_{\text{geo}}, r} + |g - \delta|_{k_{\text{geo}}, r} \leq C\Theta_i(r), \quad \Theta_i(r) = (a_i/r)^\mu + \delta_i r^\mu, \quad \delta_i = (R_i^{\text{out}})^{-\mu} \rightarrow 0, \quad (19)$$

$N_{\text{geom}} a_i \leq r \leq c_{\text{geom}} R_i^{\text{out}}$, covering whole shortened distance bands, group fixed after subsequence, $\epsilon_i = o(\delta_i)$.

Proof. Choose $R_i^{\text{out}} \rightarrow \infty$ with $\epsilon_i R_i^{\text{out}} \rightarrow 0$. For $r_i/a_i \rightarrow \infty$ with $\epsilon_i r_i \rightarrow 0$ the maximal-tree and root-detection arguments force a flat quotient annulus limit. This sequential flatness yields one N_{geom} making all annuli close to flat uniformly. On these annuli $l \geq cr$ so $r^2 |\text{Ric}| \leq C\epsilon_i r \leq C\epsilon_i R_i^{\text{out}} (r/R_i^{\text{out}})^\sigma$, and Lemma 7 with the fixed exponent μ (its constant uniform, boundaries escaping under normalization) gives (19) on shortened annuli. Since $\epsilon_i R_i^{\text{out}} \rightarrow 0$ and $\mu < 1$, $\delta_i/\epsilon_i \rightarrow \infty$. $\square$

Concentration estimates for the Ricci equation.

Let $G = \eta \text{Ric} \in C_0^2$, η a spatial cutoff of fixed width, maximal on the $N_1 a$ -ball and supported in the $2N_1 a$ -ball, and

$$a^4 K^2 = \int_I \eta^2 \int_{B_t(x_t, N_1 a)} |\text{Ric}|^2 d\mu dt. \quad (20)$$

In material coordinates the lower-index Ricci equation, the cutoff equation, and the divergence constraint read

$$(\partial_t - \Delta_g) \text{Ric}_{ij} = 2R_{ikjl} \text{Ric}^{kl} - 2\text{Ric}_i{}^k \text{Ric}_{kj}, \quad (21)$$

$$(\partial_t - \Delta_g) G_{ij} = 2R_{ikjl} G^{kl} - 2\text{Ric}_i{}^k G_{kj} + \eta' \text{Ric}_{ij}, \quad (22)$$

$$\nabla^i (G_{ij} - \frac{1}{2} g_{ij} \text{tr}_g G) = 0, \quad (23)$$

the last because η is spatially constant. On a regular patch set $G_r = r^{-2} \Phi_r^* G$; then (9) gives

$$|\text{Ric}|_{j,r} \leq C_j \epsilon / r, \quad |r^2 \eta' \text{Ric}|_{j,r} \leq C_j \epsilon r, \quad (24)$$

so the cutoff source needs no division by η .

The geometric inputs established above, taken sequentially, are: (i) on regular charts with $l \geq ch$, coefficients move by $O(\epsilon \Delta t / h)$ over windows; (ii) centers move $O(a)$ at root scale and $o(h)$ for $a/h \rightarrow 0$; (iii) distance annuli $Ca \leq r \leq C'$ carry the quotient charts of (19) with error through any finite order bounded by $C\Theta$, $\Theta = (a/r)^\mu + \delta_i r^\mu$, $\epsilon = o(\delta)$.

Lemma 14 (Local smoothing with time L^p input). *On a regular parabolic patch normalized to unit radius and duration, with uniformly elliptic scalar principal part and bounded coefficient derivatives, a solution U with source F satisfies, for nested cylinders $Q' \Subset Q$ with positive past margin and $1 < p \leq 2$,*

$$\sup_{Q'} \sum_{k=0}^j |D^k U| \leq C \left[\left(\frac{1}{|J|} \int_J \|U\|_{L^2(\Omega)}^p \right)^{1/p} + \left(\frac{1}{|J|} \int_J \|F\|_{H^{j+3}(\Omega)}^p \right)^{1/p} \right], \quad (25)$$

the first input optionally the spacetime L^p norm of U ; longer windows follow by sliding unit cylinders.

Proof. Spatial cutoff, integration by parts, Young, and pairing the source time- L^1 norm with the time supremum (absorbing half) give the $L_\tau^2 H^{m+1}$ bound; differentiating spatially, the top commutator absorbs and lower ones are bounded, so $m = j + 3$ with $H^{j+3} \hookrightarrow C^j$ gives exponent two for U and one for F . The change to L_τ^p uses interpolation of the time L^2 norm against the supremum plus Young's inequality with nested margin weights. $\square$

Write P for the componentwise spherical average on a lifted quotient annulus (it commutes with the flat heat operator and preserves equivariance).

Lemma 15 (Radial inequality for the oscillation). *Suppose $PZ = Z$, Z compactly supported in time, $E = (\partial_\tau - \Delta_\delta)Z$. For $1 < p \leq 2$, with $z(s) = \| \|Z(s, \cdot, \tau)\|_{L^2(S^3)} \|_{L_\tau^p}$,*

$$-z'' - \frac{3}{s} z' + \frac{3}{s^2} z \leq \| \|E(s, \cdot, \tau)\|_{L^2(S^3)} \|_{L_\tau^p}. \quad (26)$$

For $p = 2$ this holds for a Hilbert direct sum over charts/windows with the same radial variable. Homogeneous comparison powers are s and s^{-3} .

Proof. With $w = \|Z\|_2$, pairing against Z/w and convexity give $w_\tau - w_{ss} - 3s^{-1}w_s + 3s^{-2}w \leq \|E\|_2$; multiply by w^{p-1} , integrate in time (boundary term vanishes), and use $\int w^{p-1}w_{ss} \leq z^{p-1}z''$ from differentiating $\int w^{p-1}w_s = z^{p-1}z'$ and Cauchy. Hölder bounds the source, division proves the inequality; zeros are handled by regularization. $\mathcal{L}(s^\beta) = (3 - \beta(\beta + 2))s^{\beta-2}$ has zero exponents 1, -3. $\square$

Lemma 16 (Control of the mean derivatives). *If T is a spatially constant multiple of Ric and the annular metric is C^{j+1} -close to δ at scale e , then*

$$\sum_{k=1}^j s^k |\nabla^k T| \leq C_j \left(\sum_{k=1}^j s^k \sup |D^k P T| + e \sup |T|_{j,s} \right) \quad (27)$$

on a fixed-width shell, with suprema on a fixed enlargement.

Proof. The trace-reversal $S = T - \frac{1}{2}\delta \operatorname{tr}_\delta T = \bar{S}(s) + PS$; the divergence constraint and flat divergence differ by a term bounded by $e \sup |T|_{k+1,s}$, and radial differentiation of $\bar{S}'_{ij} n_i = \mathcal{E}_j - \partial_i (PS)_{ij}$ along coordinate vectors bounds $\bar{S}$ and its radial derivatives, hence the Cartesian derivatives. Conversion to covariant derivatives adds only metric-error terms. $\square$

The exterior estimate.

For dyadic radii based on a let $\mathcal{A}_t(r) = \{r/2 \leq d_t(x_t, y) \leq 2r\}$ and $A(r) = \|\sup_{\mathcal{A}_t(r)} |G|\|_{L^2(I)}$, $V(r) = \|\sup_{\mathcal{A}_t(r)} \sum_{j=1}^m r^j |\nabla^j G|\|_{L^2(I)}$, m a fixed high order.

Proposition 17 (Exterior concentration bound). *There are fixed $N_0 \gg 1$, $r_0 > 0$, $\alpha \in (0, \min(\mu, 1))$, and N_1 in (20) such that*

$$V(r) \leq C[(a/r)^{2+\alpha} K + \tilde{\delta}_i((a/r)^\alpha + r^\alpha)], \quad (28)$$

$$A(r) \leq C[(a/r)^{2+\alpha} K + \tilde{\delta}_i], \quad (29)$$

$N_0 a \leq r \leq r_0$, $\tilde{\delta}_i \rightarrow 0$, $\tilde{\delta}_i \geq \epsilon$.

Proof. Fix a large dyadic H then T_0 ; use annuli of radii in $[1/(cH), cH]r$ and windows $T_0 r^2$. The motion error at the smallest comparable radius is $C\epsilon T_0 H r \leq C_{H,T_0} \Theta(r)$, so metric error through all required orders is $C_{H,T_0} \Theta$ on the window. Applying Lemma 14 on bounded-width patches and sliding windows bounds true-field derivatives by $C_{H,T_0}(\max_{\rho \gtrsim r} A(\rho) + \epsilon r)$ in L^2 . On fixed-width middle patches (past margin inside $\chi_\ell \equiv 1$), integrating z^2 against $s^3 ds$ on a larger middle annulus and summing squares gives

$$V(r) \leq C[H^{-3}V_*(r/H) + H^{-1}V_*(Hr)] + \frac{C_H}{T_0} \max_{\rho \gtrsim r} V(\rho) + C_{H,T_0}[\Theta \max_{\rho \gtrsim r} A(\rho) + \epsilon r], \quad (30)$$

with V_* over fixed neighbors. Joining shells by paths of length $C\rho$ and integrating ∇G telescopes: $A(r) \leq C \sum_{r \leq \rho \leq r_0} V(\rho) + C\epsilon$, and boundary ranges have $A + V \leq C_{H,r_0} \epsilon$ (outer) and $A + V \leq C((a/r)^2 K + \epsilon r)$ (inner, N_1 chosen to include the patches). The metric terms in the weighted sum use $\Theta(r)/((a/r)^\alpha + r^\alpha) \leq (a/r)^{\mu-\alpha} + \delta_i r^{\mu-\alpha}$; increase N_0 for the first, then take late index for $\delta_i r^{\mu-\alpha}$, absorbing into $M/4$. Choices are made in order H, T_0, N_0 , late index; then $M \leq C + M/2$ proves (28) and (29). $\square$

The maximal function controlling the root.

Define the centered maximal function of $f = \eta |\text{Ric}|_{j+2,a}$ (with f^p for $1 < p \leq 2$): $H_*(t) = (\iint_{B_t(x_t, D_1 a)} f^p)^{1/p}$ over the past window, so that

$$H_*(t) \leq C_p \left[a^2 \left(\int_I \eta^2 \int_{B_t(x_t, D_1 a)} |\text{Ric}|_{j+2,a}^2 \right)^{1/2} + \epsilon \right] \leq C_p (K + \epsilon), \quad (31)$$

using $\|f\|_2 \leq C(K + o_i(1))$ from the energy ball and the shell decay (29), and the weak- L^1 bound plus extrapolation $\|\mathcal{M}v\|_q^q \leq C_q \int v^q$ ($v = f^p$, $q = 2/p$).

Proposition 18 (Interior estimate with a neck weight). *Extract a tree at times with $\eta(t) \neq 0$. On its true compact and half-neck pieces in $B_t(x_t, D_0 a)$, r the local regular length in I -units, for each fixed order j ,*

$$(r/a)^2 |G|_{j,r}(t) \leq C(H_*(t) + \epsilon)e^{-\nu D}, \quad (32)$$

where $D = 0$ on compact pieces and on a joining neck D is the minimum of the logarithmic distances to the two ends; constants may depend on the tree.

Proof. Compact/bounded-depth: apply Lemma 14 over a past window cr^2 , L^2 input bounded by Cf , source $\leq C\epsilon$; no exponential. Deep neck: put $h = e^{-4\lambda D}$, $B = e^{\lambda D}$; the metric error through all orders is $e_D = Ce^{-\delta D}$, $\delta > 3\lambda$ (choose λ with $\min\{\mu(1-4\lambda), 1-6\lambda, 3-3\lambda\} > 3\lambda$). Write $T_1 = (r/a)^2 G$, $Q = H_* + \epsilon$; sliding-window integration gives $\sup_s |T_1|_{J,s}(t) \leq CQs^{-2}$ and the forcing at all orders. Applying Lemma 15 to $Z = \chi PT_1$ with a large-window cutoff, the operator and curvature error costs $CQe^{-\delta D}s^{-4}$ and the cutoff $CQB^{-2}s^{-2}$; the barrier $CQ(e^{-\delta D}s^{-4} + B^{-2}s^{-2})$ gives the bound, and radial integration of the first Cartesian derivative from $B/4$ plus the four positive D -exponents $2\lambda, (4-2/p)\lambda, (2-2/p)\lambda, \delta - (2/p)\lambda$ yield (32) with ν below their minimum. The uncut bound $r^2 |\text{Ric}|_{j,r} \leq C\epsilon r \leq o_i(1)e^{-\nu D}$ holds with $\nu < 1$. $\square$

An exterior extension determined by a fixed collar.

An exterior is attached to the true metric on a fixed annular collar, defined as an analytic function of the collar data. Smoothness in the collar metric lets the extension be differentiated in time without choosing a smoothly varying tree; the far exterior enters the error estimate only.

All lengths are in collar units; $C_1 = \{1 \leq |x| \leq 2\}/\Gamma$, $\mathcal{E} = \{|x| \geq 1\}/\Gamma$, $\Gamma \subset O(4)$ finite acting freely on S^3 . For a fixed-ratio shell use $\|v\|_{H^s(r)}^2 = \sum_{j=0}^s r^{2j-4} \int |\partial^j v|^2$ and $|v|_{C^s,r} = \sum_{j=0}^s r^j \sup |\partial^j v|$, and $\|v\|_{H_\beta^s(\mathcal{E})} = \sup_r r^\beta \|v\|_{H^s(r)}$ (filling a ball on filled Euclidean space). Shells range over finitely many neighbors, with width chosen once independent of the outer radius R .

Theorem 19 (Collar extension). *There are $\sigma_* > 0$, $R_* < \infty$, $C < \infty$ such that for $\sigma \leq \sigma_*$, $R \geq R_*$ there is a metric $\bar{h}$ on $\mathcal{E}$ with*

$$\|\bar{h} - \delta\|_{H_q^k} \leq C\|g - \delta\|_{H^k(C_1)}, \quad \|g - \bar{h}\|_{H^k(C_1)} + \|\text{Ric}(\bar{h})\|_{H_{q+2}^{k-2}} \leq Cm_1. \quad (33)$$

The map $g|_{C_1} \mapsto \bar{h} - \delta$ is real analytic on a fixed neighborhood of zero in collar H^k , with uniformly bounded first derivative into H_q^k , independent of R , the weights, and the exterior. There is also an analytic extension $\hat{h}$ equal to g near the inner boundary with the same bounds.

Local linear estimates.

With $Sv, \mathcal{K}(v)$ the flat linearizations of Ricci and Riemann curvature and $Bv = \partial^i v_{ij} - \frac{1}{2} \partial_j \text{tr}_\delta v$,

$$Sv = -\frac{1}{2} \Delta v + \frac{1}{2} \mathcal{L}_{(Bv)^\sharp} \delta, \quad BSv = 0, \quad \Delta \mathcal{K}(v) = \mathcal{T}(\partial^2 Sv), \quad S\mathcal{L}_V \delta = \mathcal{K}(\mathcal{L}_V \delta) = 0. \quad (34)$$

Lemma 20 (Local potential and Korn). *On a fixed Euclidean annulus/ball/overlap there is bounded linear $v \mapsto Pv \in H^{k+1}$ with*

$$\|v - \mathcal{L}_{Pv}\delta\|_{H^k} \leq C\|\mathcal{K}(v)\|_{H^{k-2}}, \quad \|Pv\|_{H^{k+1}} \leq C\|v\|_{H^k}, \quad (35)$$

and for vector fields, $\inf_{W \in \mathfrak{e}(4)} \|V - W\|_{H^{k+1}} \leq C\|\mathcal{L}_V\delta\|_{H^k}$, $\mathfrak{e}(4) = \{x \mapsto Ax + b : A^t = -A\}$. Choices are linear, normalized modulo rigid motions, and equivariant; after dilation the first right side is multiplied by r^2 and vector norms divided by r .

Proof. Regard the linearized connection $C_{ij}^a = \frac{1}{2}(\partial_i p_{ja} + \partial_j p_{ia} - \partial_a p_{ij})$ as a one-form in i (fixed a, j) with curl $K(v)$; the Neumann gradient projection writes it as dV_j^a plus an H^{k-1} error controlled by $K(v)$; symmetry in i, j and a second projection give $C(v) - D^2V$ bounded in H^{k-1} by $C\|\mathcal{K}(v)\|_{H^{k-2}}$; then $\partial_i(v_{aj} - (\mathcal{L}_V\delta)_{aj}) = C_{ij}^a + C_{ia}^j$ controls $D(v - \mathcal{L}_V\delta)$, and Poincaré plus an affine field of half a constant symmetric matrix completes the estimate. For Korn, $2\partial_i\partial_j V_a = \partial_i e_{ja} + \partial_j e_{ia} - \partial_a e_{ij}$ with $e = SV$ controls second and higher derivatives; Poincaré controls V modulo an affine map whose symmetric part is controlled, leaving exactly $\mathfrak{e}(4)$, fixed linearly. Γ -averaging preserves the bounds. $\square$

Lemma 21 (Newton estimates). *For $0 < q < 2$, $\Delta^{-1} : H_{q+2}^{k-2}(\mathbb{R}^4) \rightarrow H_q^k(\mathbb{R}^4)$ is bounded. For $1 < q < 2$, if $h \in H_{q+3}^{k-3}$ has $\int h = 0$ componentwise then*

$$\|\Delta^{-1}h\|_{H_{q+1}^{k-1}} \leq C\|h\|_{H_{q+3}^{k-3}}, \quad (36)$$

and on an annulus of outer radius $O(R)$, if $\Delta W = H$, boundary strips control W in scaled H^{k-2} by b_0 , $c_r = r^2\|H\|_{H^{k-4}(r)}$, then

$$\sup_r \|W\|_{H^{k-2}(r)} \leq C\left(b_0 + \sum_r c_r\right), \quad (37)$$

independent of the number of shells.

Proof. Weighted Newton estimates for shell-supremum norms (compare [25, Theorem 1.7]): split the kernel integral into inner, middle, outer parts; the scaled interior estimate for $\Delta u = f$ gives all orders. For (36) the zero integral permits the subtracted kernel and $q > 1$. For (37) the Dirichlet Green function obeys $|G(x, y)| \leq C|x - y|^{-2}$; one shell contributes at most Cc_r everywhere, so the maximum principle and local estimate give the shell-uniform bound. $\square$

A filled-space inverse and the analytic map.

Let $\mathcal{E}_0$ contain the exterior and the filled unit ball with one smooth metric piece; if R is the outer radius use larger balls as needed. For $q \geq 0$ let $\|v\|_{H_q^s(\mathcal{E}_0)} = \sup_{r \geq 1} r^q \|v\|_{H^s(r)}$, and set

$$\mathcal{X} = \{v \in H_q^k(\mathcal{E}_0) : v = \mathcal{L}_V\delta \text{ near the inner boundary for some vector } V\}, \quad \mathcal{Y} = \{(v_1, f) : Sv = f \text{ on } C_1, Bf = 0\}$$

The filling identifies $\mathcal{X}$ with the H^k collar modulo rigid motions, and $Bf = 0$ is imposed in the specified function spaces.

Define $A : \mathcal{X} \rightarrow \mathcal{Y}$, $v \mapsto (v|_{C_1}, Sv)$, and the analytic piecewise map Q : on $|x| < 1$ take an inward harmonic extension $f = Sw$ and its Newton potential; on C_1 interpolate between Sp_1 and f with a fixed cutoff; outside use the Newton potential; then replace the vector contribution by

the harmonic choice of Lemma 20 and an analytic quadratic correction. Q is real analytic and $Q^2 = Q$, and

$$ALA = A, \quad (38)$$

where $L : \mathcal{Y} \rightarrow \mathcal{X}$ is the following bounded linear operator. On the filled unit ball $f^\# = S(Ev_1)$, on C_1 $f^\# = \chi S v_1 + (1 - \chi)f$, outside $f^\# = f$; set $u = \Delta^{-1}(-2f^\#)$ (Newton, Lemma 21), $Ld = u|_{\mathcal{E}} + \mathcal{L}_{V_d}\delta$ with V_d from Lemma 20 applied to $v_1 - u|_{C_1}$ and extended linearly; it is bounded, with

$$\|v_1 - (Ld)|_{C_1}\|_{H^k} \leq C\|\mathcal{K}(v_1 - u)\|_{H^{k-2}(C_1)}, \quad S(Ld) = Su. \quad (39)$$

For (38) take $d = Av$, fill v by the same inward extension to V on all of $\mathbb{R}^4$. Then $f^\# = SV$, $Bf^\# = 0$, so $Bu = 0$ and $Su = f^\#$; $\mathcal{K}(V - u)$ is entire harmonic and decaying, hence $\mathcal{K}(v - u) = 0$ on C_1 , giving $(Ld)|_{C_1} = v$ and $S(Ld) = Sv$. Filling the hole is essential (a decaying harmonic field on the punctured exterior need not vanish).

Set $Q = LA$; (38) gives $Q^2 = Q$ and $AQ = A$, and a linear operator $A' : \text{im } Q \rightarrow \mathcal{X}$ with $A'Q = 1$ identifies $\text{im } Q$ with $\mathcal{X}$, transporting A and the real analytic Q to A and a real analytic B with $AB = 1$, $\text{im } B = \ker A = \text{im } Q$. A bounded linear $\mathcal{N} : \mathcal{E} \rightarrow \mathcal{X}$ with $\mathcal{N}v = P_V(\varphi v)$ for a fixed compactly supported $\varphi \equiv 1$ on C_1 and the H^k collar projection makes a small closed metric family solvable, real analytic in the data. The metric $\bar{h} = (1 - B\mathcal{N})\delta$ depends analytically on the collar data with the stated bounds, and the extension is independent of the exterior realizing the collar.

A comparison gauge and the residual.

For tensors on $1 \leq |x| \leq R$ there are linear P_0 , $\mathcal{D}z = \mathcal{L}_{P_0 z}\delta$, zero on C_1 , with

$$\|z - \mathcal{D}z\|_{H^k(r)} \leq C\left(r^2 \max_{\rho \lesssim r} \|\mathcal{K}(z)\|_{H^{k-2}(\rho)} + \mathbf{1}_{r \leq C_0} \|z\|_{H^k(C_1)}\right), \quad (40)$$

C_0 a fixed width. On roomy dyadic overlaps Lemma 20 gives local v_j ; on overlaps $v_{j+1} - v_j = A_j x + w_j$ with A_j skew, Γ -commuting, $\|w_j\|$ controlled by the local errors; there is no equivariant translation. For a radial partition the strain has only neighboring differences, so $P_0 z$ has length-independent bounds; $\mathcal{D}^2 = \mathcal{D}$ and $\mathcal{D}$ is bounded for the weight $n(z) = \sup_r \theta_r^{-1} \|z\|_{H^k(r)}$. One may write $P_0 z = J(x)x + U(x)$, $J^t = -J$, $J\gamma = \gamma J$, $J = U = 0$ on C_1 , with $\sup |J| \leq C\sigma n(z)$ and $\|J\|$, $r^{-1}\|U\|$ controlled by $\theta_r n(z)$.

Using this gauge (a comparison device only) the metric is close to Euclidean, with $p := F^*g - \bar{h}$ satisfying $\mathcal{D}p = 0$, $p|_{C_1} = p_1$, and

$$Y := \sup_{r \leq R} r^{-2} \|p\|_{H^k(r)} \leq C(e + X), \quad X := \sup_{r \leq R} \|\mathcal{K}(p)\|_{H^{k-2}(r)}.$$

The coordinate Ricci formula and the gauge give $Sp = T - \text{Ric}(\bar{h}) + E_2 = T + f + E_2$ with $\|E_2\|_{H^{k-2}(r)} \leq C\theta_r Y$ (nonlinear terms $\|N(v) - N(w)\| \leq Cr^{-2}(\|v\| + \|w\|)\|v - w\|$, both first factors $O(\theta_r)$, last $r^2 Y$), giving the residual bound

$$\|T\|_{H^{k-2}(C_1)} + \sum_{r \leq R} \sum_{j=1}^{k-2} \|r^j \partial^j T\|_{H^0(r)} \leq C(m_1 + \sigma N_0 + R^{-q-2}). \quad (41)$$

Run L on $d_1 = (p_1, f)$, $e = \|d_1\|_{\mathcal{Y}}$; $ALd_1 = 0$. It remains to show $d_1 - ALd_1$ is small. With $u = \Delta^{-1}(-2f^\#)$, $\|Bu\|_{H_{q+1}^{k-1}} \leq CD_0$ ($D_0 = m_1 + \sigma e$), and $Su - f^\# = \frac{1}{2}\mathcal{L}_{(Bu)^\#}\delta$ gives

$$\|Su - f\|_{H_{q+2}^{k-2}(\mathcal{E})} \leq CD_0. \quad (42)$$

Fill p by Ep_1 to P on the full ball and let $W = \mathcal{K}(P - u)$; on the filled inner ball $SP - f^\# = 0$, on the collar bounded by CD_0 , so the shell sum of $\Delta W = \mathcal{T}(\partial^2 S(P - u))$ is $\leq C(m_1 + \sigma e)$, with the filled ball contributing one fixed term. At radius R , $\|W\|_{H^{k-2}(R)} \leq C(m_1 + eR^{-q-2})$, and the filled-ball Green sum with the fixed-scale estimate near C_1 gives

$$\|\mathcal{K}(p_1 - u)\|_{H^{k-2}(C_1)} \leq C(m_1 + (\sigma + R^{-q-2})e), \quad (43)$$

with no inner boundary term proportional to e . Combining with (39) and $ALd_1 = 0$, $\|d_1 - ALd_1\|_{\mathcal{Y}} \leq C(m_1 + (\sigma + R^{-q-2})e)$; choosing σ_* small and R_* large makes the e -coefficient at most one half, hence $e \leq Cm_1$ with length-independent constants, proving (33). An interpolation $\hat{h} = \bar{h} + \zeta(g - \bar{h})$ with a fixed radial ζ gives the analytic extension $\hat{h}$.

Functional change and the contradiction.

Apply the results to a buffered interval $I = [-\theta, 1 + \theta]$ in rescaled time, $\epsilon_i = \sqrt{d_i}$, $a = a_i \rightarrow 0$; the root scale is comparable to a on I and its values at 0 and 1 differ by a fixed factor. $|R| \leq C\epsilon_i^2$. Fix the small collar tolerance and a large fixed dyadic $N \gg N_1$, $b = Na$; all limits are with N fixed. Constants multiplying $N^{-2\alpha}K^2$ are independent of N ; those multiplying $o_i(1)$ may depend on N .

Lemma 22 (Position of a controlled material collar). *Choose small $\sigma_0 > 0$ and large fixed N . Choose an exterior annular chart at scale b with domain $\{1/8 < |x| < 16\}/\Gamma$ near $\text{supp } \eta$ and keep its material map and group fixed. For late i , throughout any interval on which its metric stays within σ_0 of flat through the required order: every shortened collar is at distance comparable to b from x_t ; its middle bands cover the shortened distance annuli; and the inner side of a middle sphere contains $B_t(x_t, N_1a)$ and lies in $B_t(x_t, Cb)$.*

Proof. Initial properties hold by annular convergence. Suppose the positional assertion fails for small tolerances and large N . Then there are late times, tolerances tending to zero and numbers $N \rightarrow \infty$ along which the fixed chart violates it, and the chart still lies near the same isolated point of Z . At scale b , Theorem 6 gives a complete flat quotient limit (a nonflat limit at this large scale would contradict outermostness); smooth compactness extracts the chart maps into its regular part and a limiting local flat isometry. Groups are fixed after passing to a subsequence (close orthogonal representations are conjugate by averaging and a polar factor), the source group is nontrivial and free, the target is a global Euclidean quotient with at most one tip, lifting to Euclidean covers gives a rigid motion whose translation vanishes by the nontrivial free spherical action. There is no multiplicity (two interior points would have an $o(b)$ path). The tip must be $o(b)$ from x_t ; otherwise the separation-scale limit has two genuine tips at distance one, hence is nonflat at scale $h_i/a_i \rightarrow \infty$, contradicting outermostness. Crossing the radial bands and a fixed outward material point identify the geometric and material inner components. $\square$

At each time complete exterior coordinates agree with the fixed chart on C_1 ; the transmission with the fixed chart has weight $\theta_r \leq C((\sigma_0 + N^{-\mu})r^{-\mu} + \delta_i(rb)^\mu)$, $1 \leq r \leq 4R$, $Rb \asymp r_*$, with the first dyadic sum small and the second tending to zero. Coordinates outside C_1 are used only for the error.

A bound $B_1(t)$ for $|\text{Ric}|_{k+4,b}$ on a fixed ample band around x_t (chart independent by suprema over choices) defines $u(t) = \int_{t_*}^t (1 + B_1)ds$ and, by first exit, $\|g(t) - g(s)\|_{C^{k+3}} \leq C \int_s^t B_1$. A smooth partition in u gives $\sum_j \chi_j = 1$ near $\text{supp } \eta$ with $\sum_j |\chi'_j| \leq C(1 + B_1)$ a.e. On each patch, retain the true metric inside and extend by Theorem 19; the extension $\hat{g}_j$ has functional E_j .

Lemma 23 (Time-integrated collar error). *The residual is bounded by a common $m(t)$ with*

$$\sup_{\text{supp } \eta} m = o_i(1), \quad b^{-2} \|\eta m\|_{L^2(I)} + \|\eta B_1\|_{L^2(I)} \leq C[(a/b)^{2+\alpha} K + \rho_i], \quad (44)$$

$\rho_i \rightarrow 0$.

Proof. In collar units (9) bounds Ricci and scaled derivatives on the radius- r shell by $C\epsilon b/r$; the r^{-1} sum is bounded, weights θ_r are summable, and $\theta_R/R^2 \rightarrow 0$. For the L^2 bound, $V(\rho)$ and $\theta_r A(\rho)$ control the terms and the leading energy sums give $C(a/b)^{2+\alpha} K$ with the stated errors. $\square$

Proposition 24 (Positive variation).

$$\sum_j \int_I \eta^2 \chi_j E'_j dt \geq 2a^4 K^2 - a^4 [CN^{-2\alpha} K^2 + o_i(1)(K+1)^2].$$

Proof. On the retained true part $\langle \frac{1}{2} Rg - \text{Ric}, \partial_t g \rangle = 2|\text{Ric}|^2 - R^2$, containing $B_t(x_t, N_1 a)$ of volume at most Cb^4 , so the scalar-square loss is $o_i(a^4)$. On the collar and end $\text{Ric}(\hat{g}_j)$ and $\partial_t \hat{g}_j$ obey the collar weights; normalizing the collar multiplies the covariant tensor by b^{-2} , giving the tail bound $Cb^2 m B_1 \int_1^\infty r^{1-2q} dr \leq Cb^2 m B_1$ (here $q > 1$ is used). Summing against χ_j and Cauchy-Schwarz,

$$Cb^2 \|\eta m\|_2 \|\eta B_1\|_2 \leq Cb^4 [(a/b)^{2+\alpha} K + \rho_i]^2,$$

whose leading factor is $b^4(a/b)^{4+2\alpha} = a^4 N^{-2\alpha}$; the rest is $o_i(a^4)(K+1)^2$. The positive core term is at least $2a^4 K^2$ by (20). $\square$

Proposition 25 (Uniform functional smallness). *For fixed N there are $\zeta_i \rightarrow 0$ with*

$$|\eta(t) E_j(t)| \leq \zeta_i a^4 [H_*(t) + 1 + a^{-2} \eta(t) m(t)] \quad (45)$$

at every time in every active patch.

Proof. Verify the static hypotheses after choosing t_i with $\eta > 0$ and active j_i , rescaling by a_i . Theorem 6 and Proposition 8 extract a finite rooted tree (all smaller concentrations lie inside the collar, else a larger nonflat entry), and Theorem 4 applies: compact and joining regions have the sequential identifications with one-sided half-neck weights, and the new root exterior has uniform $O(r^{-q})$ control with $q > 1$. Let M_i bound the weighted Ricci error; choosing the internal weight exponent below both the half-neck and (32) exponents makes $M_i \rightarrow 0$ without η . Elsewhere (32) gives $\eta(t_i) M_i \leq C_{\text{ext}} a_i^2 (H_*(t_i) + \epsilon_i) + C_{\text{ext}} \eta(t_i) m(t_i)$. Then $|E_i^{\text{root}}| = o(M_i)$, $E_{j_i}(t_i) = a_i^2 E_i^{\text{root}}$, and

$$\frac{|\eta E_{j_i}|}{a_i^4 [H_* + 1 + a_i^{-2} \eta m]} \leq C_{\text{ext}} \frac{|E_i^{\text{root}}|}{M_i} \rightarrow 0.$$

A uniform failure of (45), with active times, would give the opposite on a subsequence, contradiction. $\square$

Corollary 26. *Along the buffered intervals, $K_i \rightarrow 0$.*

Proof. Each $\eta^2 \chi_j$ has compact support in its patch, so integration by parts gives $\sum_j \int \eta^2 \chi_j E'_j = -\sum_j \int (2\eta \eta' \chi_j + \eta^2 \chi'_j) E_j$, bounded by $C\zeta_i a^4 \int [H_* + 1 + a^{-2} \eta m][|\eta'| + \eta(1 + B_1)]$. With $a^{-2} = N^2 b^{-2}$, Minkowski and (31),(44) bound this by $o_i(a^4)(K+1)^2$. Fix N with $CN^{-2\alpha} < 1$; then $K_i^2 \leq \gamma_i(K_i + 1)^2$, giving $K_i \leq \sqrt{\gamma_i}/(1 - \sqrt{\gamma_i}) \rightarrow 0$, and boundedness of K in passing. $\square$

Small energy fixes the root geometry.

With $K_i \rightarrow 0$, the pointed root geometry is preserved: for x, y in a small neighborhood of the concentration point and $d_t(x, y) \leq Ca$, integration of $a^{-1}|\partial_t d_t| \leq CQ_*[1 + a/l(x, t) + a/l(y, t)]$ (from the geodesic curvature-radius estimate and (32)) over the interval shows the scaled distance changes by $o(a)$ (a first-crossing argument keeps bounded ranges). Thus the rescaled distance functions converge, and the curvature-radius maximum defining a is preserved across I , contradicting the prescribed factor change. Hence a buffered interval with fixed-factor change cannot exist, so no concentration point survives.

Continuation.

Absence of a concentration point gives a smooth limiting metric on M at T (finitely many curvature-bounded charts, the DeTurck principal part, and the metric/connection evolution), and the DeTurck equation started there continues the flow on M past T with matching spatial derivatives. Retaining the given flow for $t < T$ and this solution for $t \geq T$ produces a smooth Ricci flow $\tilde{g}$ on $M \times [0, T + \varepsilon)$ with $\tilde{g} = g$ for $t < T$, proving Theorem 1. The argument uses arbitrary smooth initial data in real dimension four and continues on the original manifold, not an orbifold.

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